Lense–Thirring Effect

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Outline

- Linear Approximation of Rotating Spacetime
- Quasi Lamor precession
- Node precession of Circular orbits
- Twisted Accretion Disks

Towards a metric for rotating spacetime

Spherically Symmetric vacuum sol around a BH:

Schwarzschield metric (1915)

Vacuum Solution around a rotating BH:

Kerr metric (1963)

How do we discuss the effect of a spinning BH when an exact solution wasn't found yet?
(Lense & Thirring 1918)

Perturbation Solution via Green function

(Lorentz gauge)

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

The Green function $G(x^\sigma - y^\sigma)$ for the d'Alembertian operator $\Box$ is the solution of the wave equation in the presence of a delta-function source:

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma), \qquad (7.126)$$

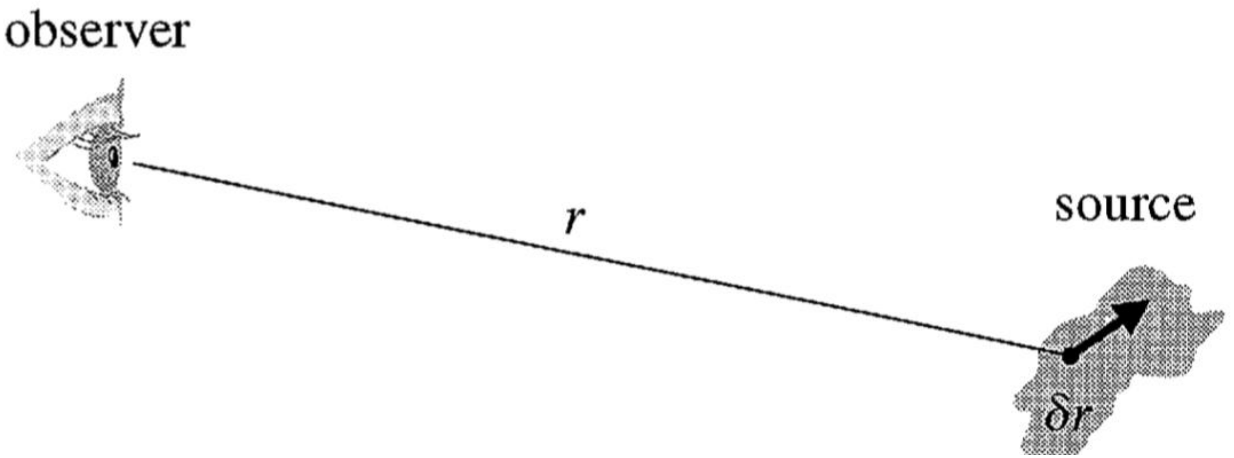
Remember The Green Function to a Laplace Operator in solving electric fields is a delta function!

setting elapsed time as t, and separating the time & space coordinates give

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int \frac{1}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y}) d^3y$$

time-independent metric & momentum tensor (with diagonal terms)

Perturbation Solution-first order



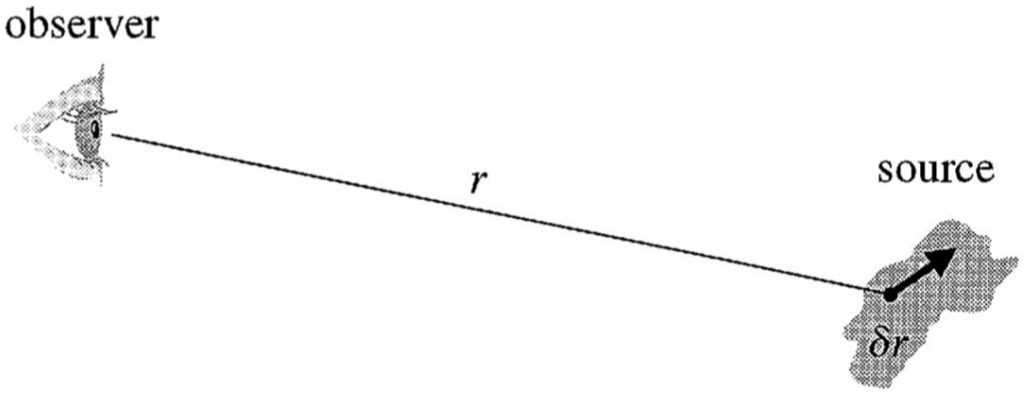
$$\bar{h}_{\mu\nu}(\mathbf{r}) = 4G\int_S \frac{1}{|\mathbf{r}-\delta\mathbf{r}|}T_{\mu\nu}(\delta\mathbf{r})d^3\delta r$$

The field at observer location (r) is determined by the energy momentum tensor integrated over a source

what approximations can be made when every bit and bob of the source is far, far away (delta r << r)?

Taylor expansion to first order in delta r/r or y/x

Perturbation Solution-first order



Compare to the simplest form of perturbation!

$$h_{00} = -2\Phi$$
$$h_{0i} = w_i$$
$$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij},$$

$$ds^2 = -(1 - 2\frac{GM}{r})dt^2 - (2\epsilon_{jkl}J^k\frac{x^l}{r^3})(dtdx^j + dx^jdt) + (1 + 2\frac{GM}{r})(dx^jdx^j)$$

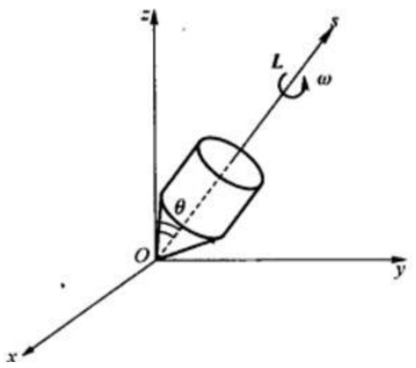
Where $M = \int T^{00}d^3x$, $J_k = \int \epsilon_{klm}x^lT^{m0}d^3x$

integrated over an entire source (small)

Corresponding to a perturbation with off-diagonal terms but without the strain

The integration tells you what these abstract values (w, \Phi) really corresponds to for the property of a source (Mass M and Angular momentum J) $\sim \rho dx^3 U^0 \vec{x} \times \vec{U}$

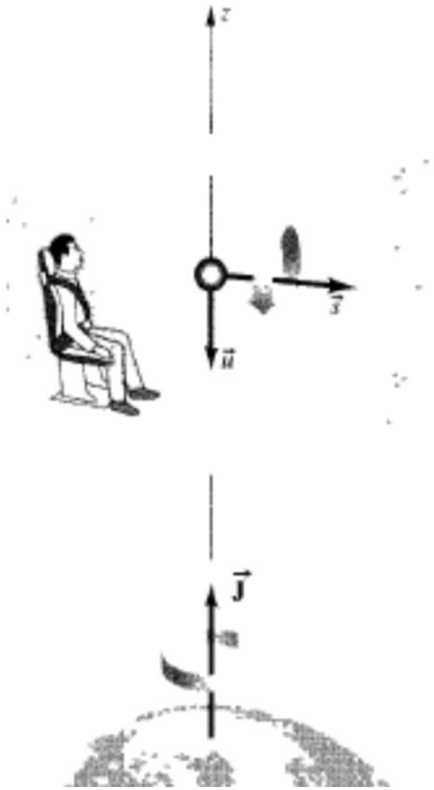
Precession of a Gyroscope (陀螺)



Very interesting and profound effect is on a free-falling gyroscope

We learned in DAXUEWULI that a normal gyroscope will precess, if gravity gives it a constant torque from center of mass

But what about a freely falling gyroscope? in the non-inertial frame, so-called ficticious force will cancel out the gravitational torque

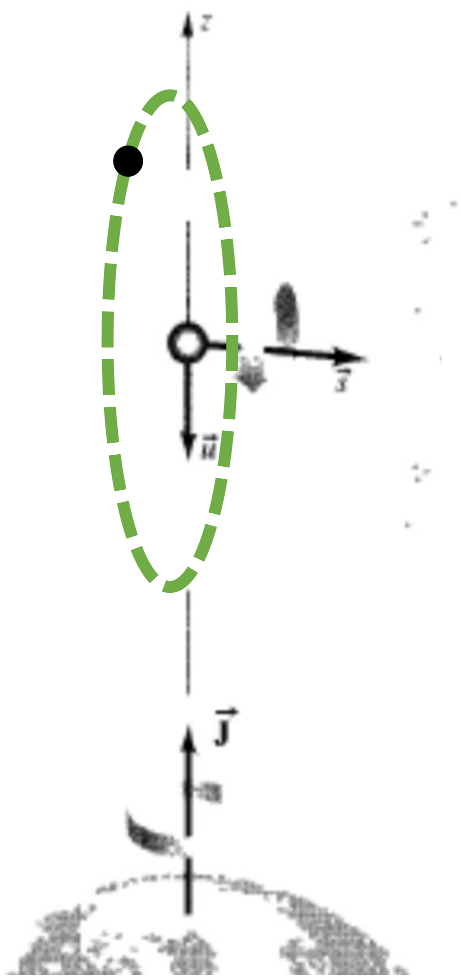


Suppose you are slowly falling towards the Earth rotating with an angular momentum s not parallel with the Earth's spin J (not very enjoyable)

will your angular momentum vector still precess?

Newtonian mechanics: No
GR: Yes (Lense & Thirring 1918)

Analogy with Lamor Precession



Let's first take one particle rotating about the spin axis (while falling) with some specific angular momentum (e.g. YOUR EYEBALL)

Caroll says its Equation of motion goes as:

$$\frac{dp^i}{dt} = E\left[G^i + (\vec{v} \times H)^i - 2(\partial_0 h_{ij})v^j - \left(\partial_{(j} h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right) v^j v^k\right].$$

time-independenct

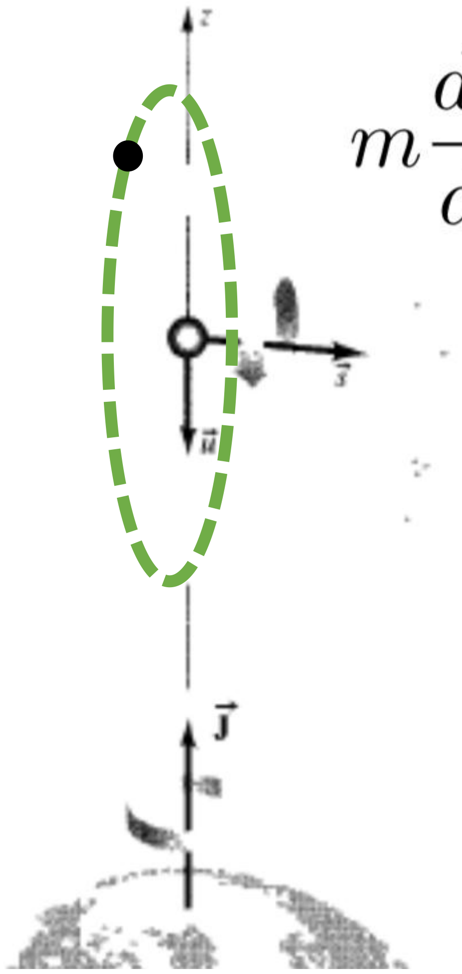
Higher order in h

Where the gravito-electric field & gravito-magnetic field is:

$$G^i \equiv -\partial_i \Phi - \partial_0 w_i$$
$$H^i \equiv (\nabla \times \vec{w})^i = \epsilon^{ijk}\partial_j w_k,$$

Analogy with Lamor Precession

compare with Lorentz force!

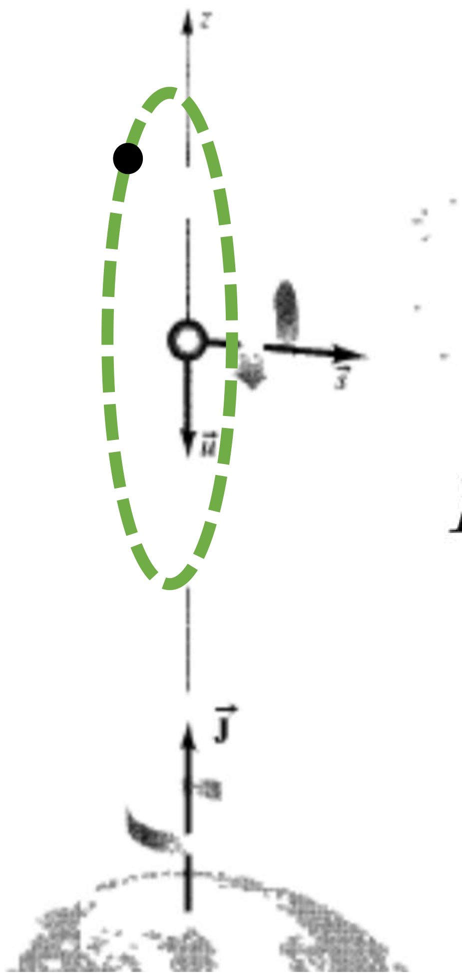


$$m\frac{d\vec{a}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \Longleftarrow m\frac{d\vec{a}}{dt} = m(\vec{G} + \vec{v} \times \vec{H})$$

What will a charged particle under Lorentz force do: Overall being accelerated by E, but the spin will rotate about the axis of B with Lamor precession rate (a known result)

$$\vec{\Omega} = \frac{q}{2m}\vec{B} \quad \xrightarrow{\text{translates into}} \quad \vec{\Omega} = \frac{1}{2}\vec{H}$$

Analogy with Lamor Precession



What is this $\vec{\Omega} = \frac{1}{2}\vec{H}$ here?

$$H^i \equiv (\nabla \times \vec{w})^i$$

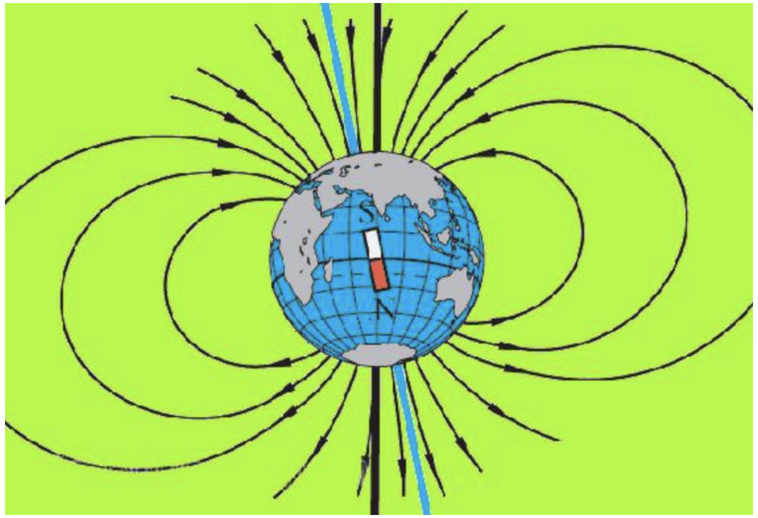
$$\vec{w} = 2G\vec{J} \times \vec{r}/r^3$$

Analogous to the vector potential A generated by magnetic DIPOLE!

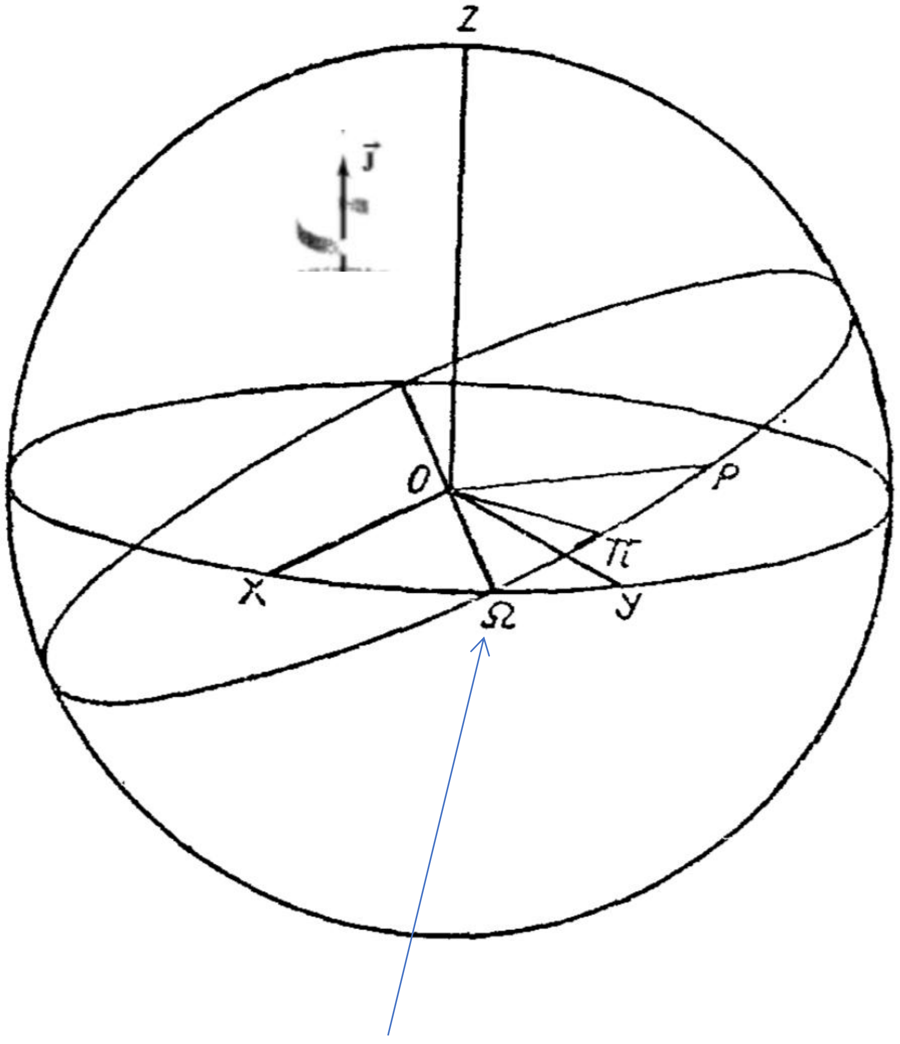
$$\vec{H} = \frac{2G}{r^3}\left[3\left(\vec{J}\cdot\vec{e}_r\right)\vec{e}_r - \vec{J}\right]$$

Analogous to the magnetic field B generated by magnetic DIPOLE!

only this one REALLY centers the south pole and the north pole



Node Precession of a Circular Orbit



Ascension Node (春分点，升交点)

Consider a large inclined (倾斜) orbit around the central body

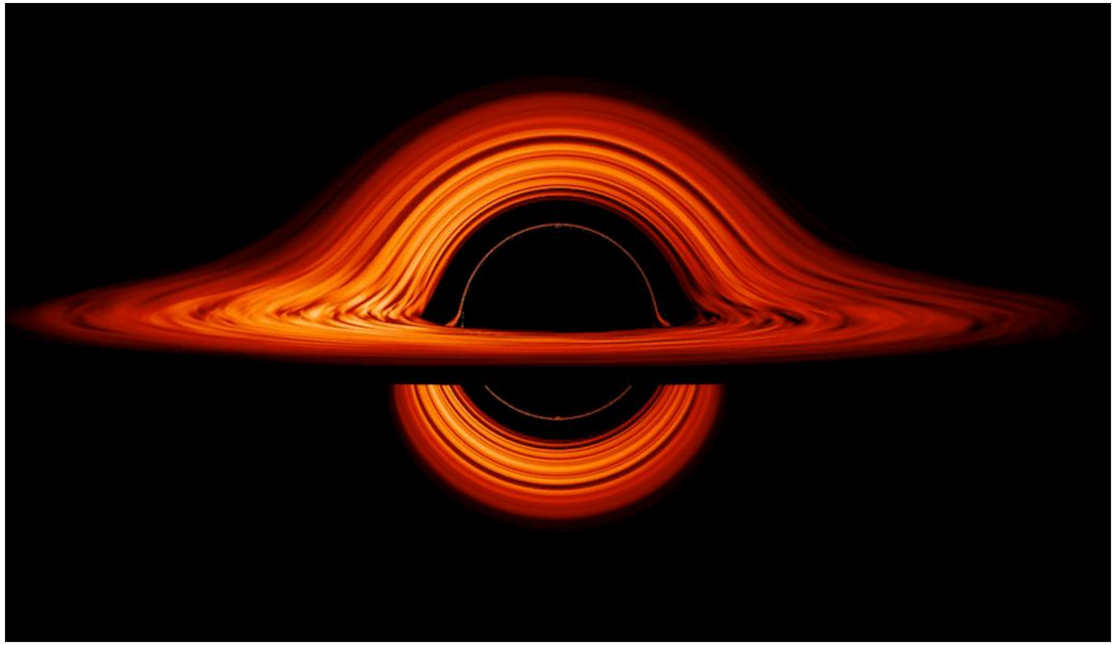
Orbital plane (轨道面) is inclined with respect to the central body's equatorial plane (赤道面), they intersect at the Ascension Node

The LT torque (different at every location and orientation!) will give an overall effect that causes the plane to precess at

$$\vec{\Omega}_{LT} = \frac{2G\vec{J}}{r^3}$$

(not trivial to calculate!)

Accretion Disk



Accretion Disks:

Matter collapsing towards a central body (protostar, protoplanet, BH, SMBH) but falls onto a mutual plane due to angular momentum conservation
Matter continues to spiral inwards, potential energy -> light and heat

How can we mathematically describe them?

Accretion Disk in the simplest form

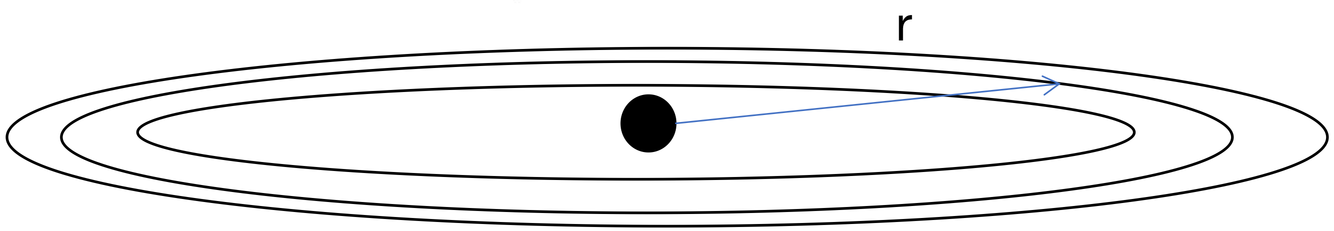
First approximation

- Totally axisymmetric, no thickness
- many many gas particles orbiting the central body at different radius, each on Keplerian orbit (but then they don't spiral inwards anymore!)

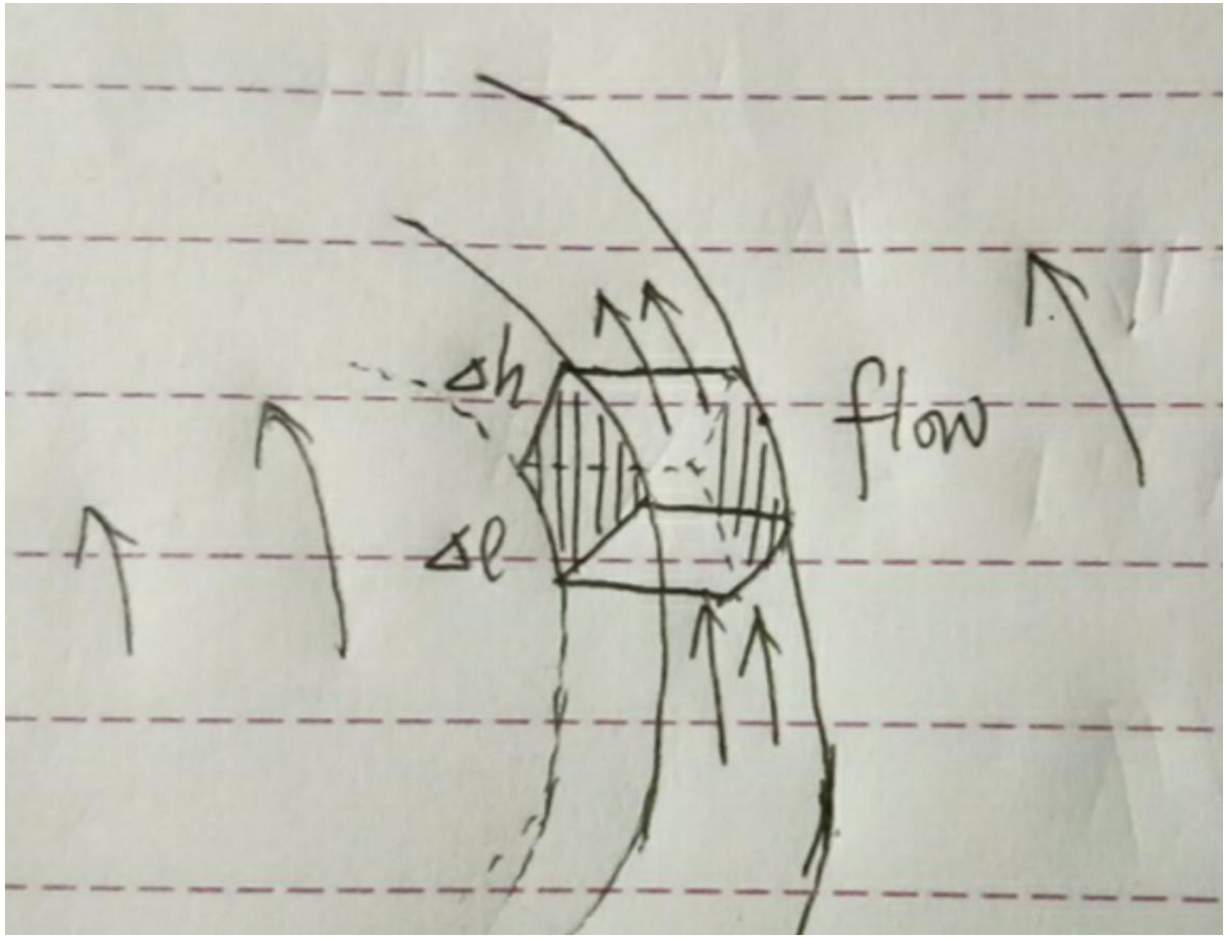
Second approximation

Due to viscous force（粘滞力）, there will be an inward velocity which can be described by a function of of r that is nearly zero (specific form does not matter!)

$$v_\phi(r) \approx \sqrt{\frac{GM}{r}}; v_r(r) \lesssim 0$$



What is viscous force?



$$F = \frac{\Delta v}{\Delta x} S\mu$$

Proportional to the shear or velocity difference between layers

Leading to energy and angular momentum dissipation and spiralling in of some materials

Inclined Accretion Disk around a BH

Consider a simple way of forming an accretion disk:

- First, infinite materials at a same very, very large radius start with fixed precession angle, but their orbits are intrinsically inclined with some angle
- Then, they continuously drift inwards one after another

without LT: they stay on the same plane with zero precession phase
with LT: all of them will experience a unique precession phase at a unique radius of r

$$\gamma_p \approx \int_0^{t(r)} \Omega_{LT} dt = 2GJ \int_\infty^r r^{-3} \frac{dr}{v_r}$$

This is the Precession Phase, not the inclination angle or the position angle, only tells you how much the total orbit (with fixed inclination) has shifted

Inclined Accretion Disk around a BH

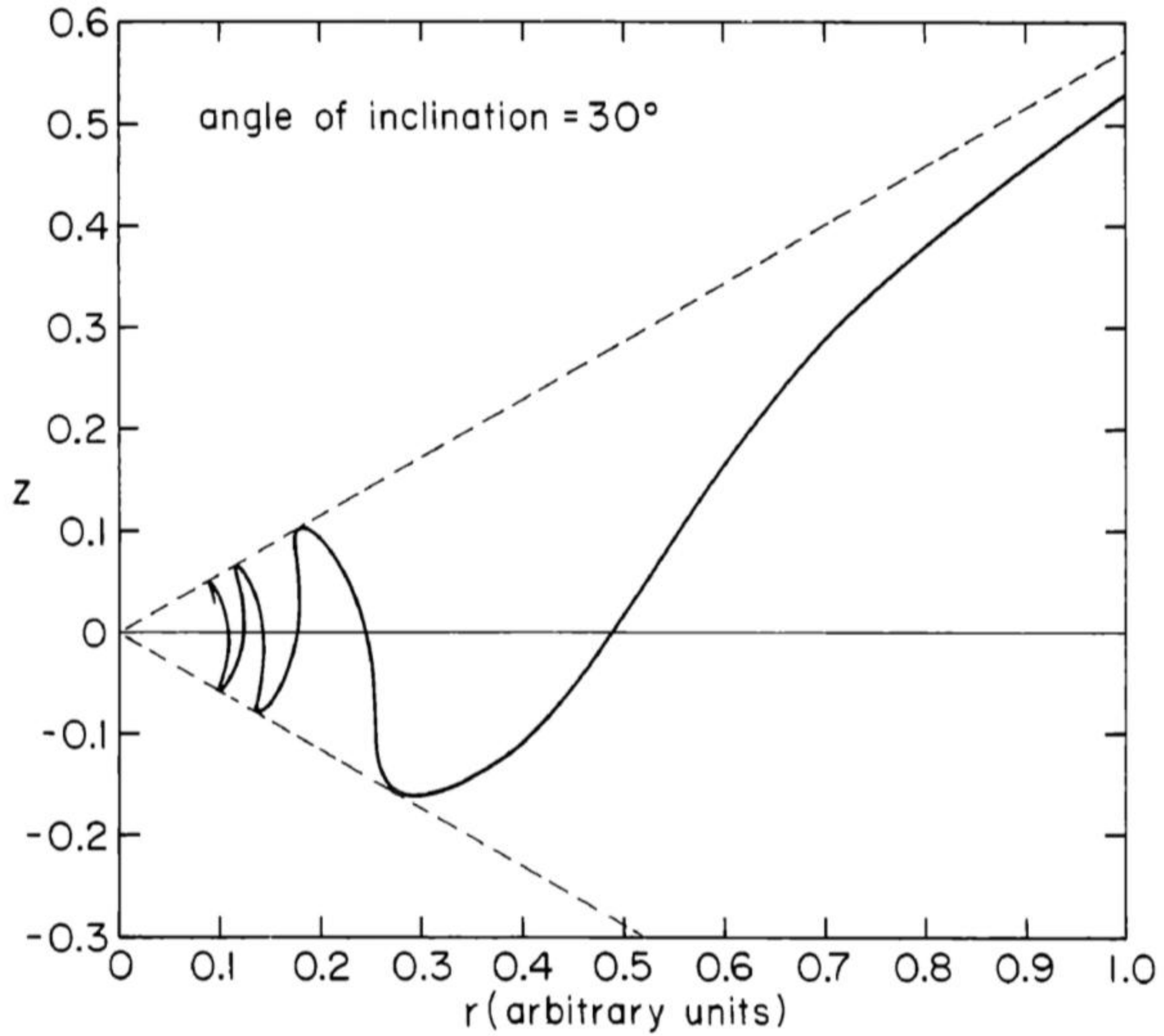


FIG. 1.—A cross section of an accretion disk which far from the black hole is tilted at an angle 30° with respect to the equatorial plane of the black hole. The units of the cylindrical coordinates r

(Bardeen & Petterson 1975)

$$v^r \approx -2.4 \times 10^{-5} \alpha^{4/5} \dot{M}_*^{2/5} M_*^{-3/5} r_*^{-2/5} .$$

A cross section of the accretion disk after some time, All the orbits are tilted 30 degrees with respect to the equtorial plane of the BH, but they "touch bottom" and "touch ceiling" at different locations so the disk is distorted!

Inclined Accretion Disk around a BH

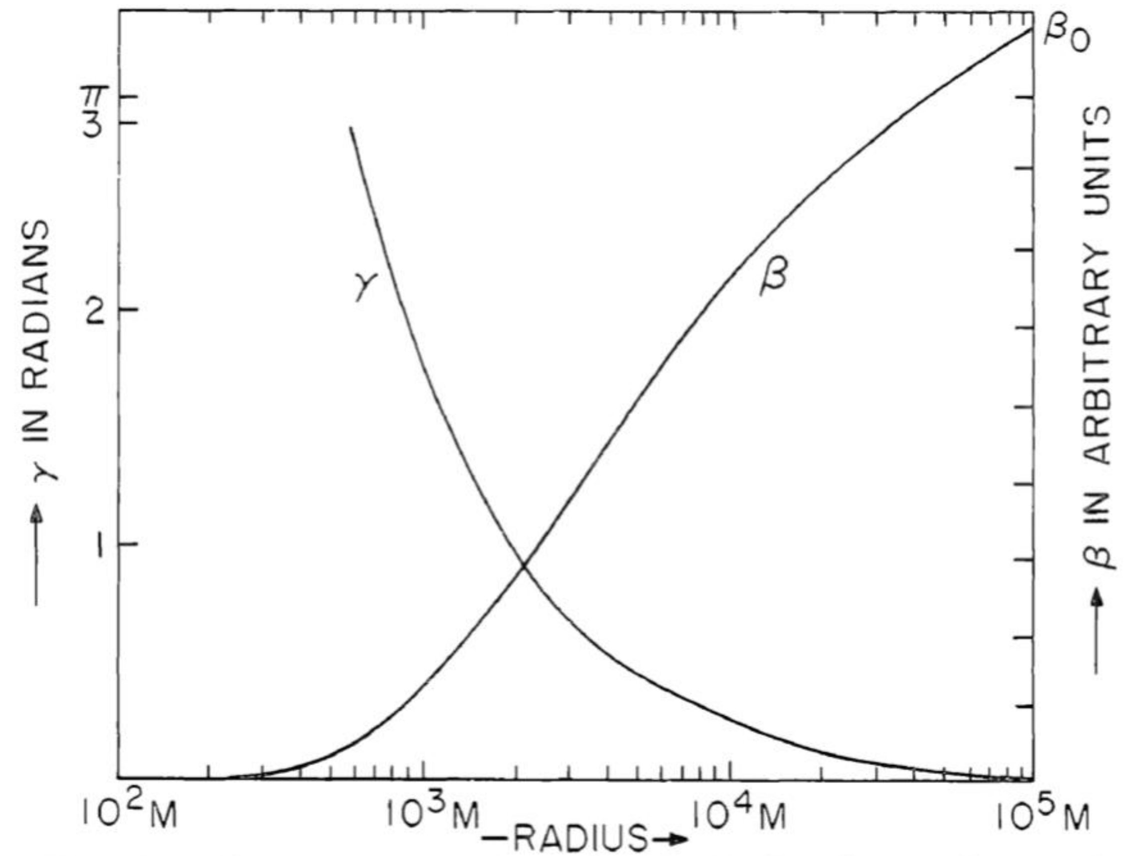
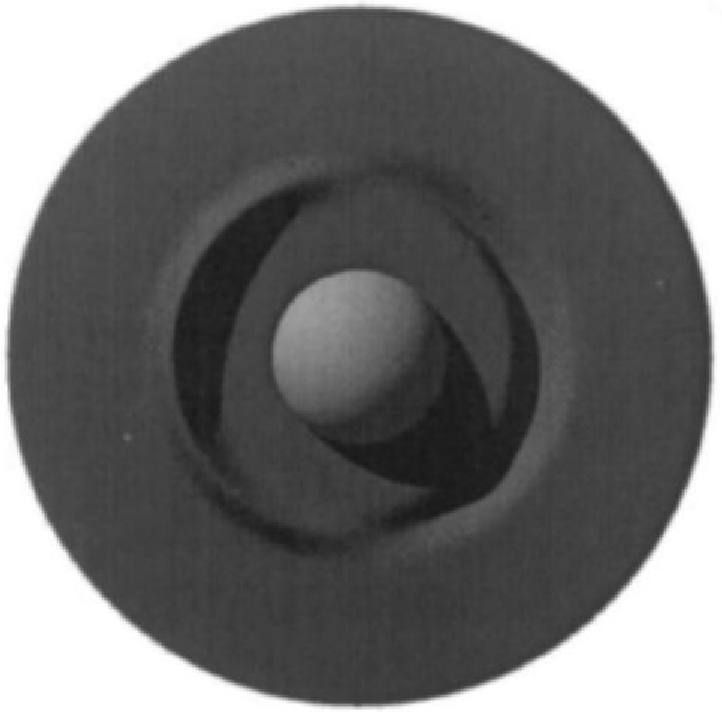


FIG. 3.—Plotted as a function of r, the inclination angle β and the total precession angle γ of a ring of radius r of an accretion disk whose outer edge is tilted at an angle β_0. The curve for β shows that at radius $500M$ the disk is almost completely relaxed into the equatorial plane of the black hole; this is a result of the presence of viscous forces.

But, since viscous torque is against any kind of shear, it will also flatten the difference in inclinations (velocity difference in z directions!)

Adding such a torque will result in change of inclination angle -> around the BH the disk connects to the midplane
which explains double-plane accretion disks



Summary

- Linear Approximation of Rotating Spacetime

$$ds^2 = -(1 - 2\frac{GM}{r})dt^2 - (2\epsilon_{jkl}J^k\frac{x^l}{r^3})(dtdx^j + dx^j dt) + (1 + 2\frac{GM}{r})(dx^j dx^j)$$

- Quasi-Lamor precession

$$\vec{\Omega} = \frac{1}{2}\vec{H} \qquad \vec{H} = \frac{2G}{r^3}\left[3\left(\vec{J}\cdot\vec{e}_r\right)\vec{e}_r - \vec{J}\right]$$

- Node precession of Circular orbits

$$\vec{\Omega_{LT}} = \frac{2G\vec{J}}{r^3}$$

- Twisted Accretion Disks

Free LT precession causes an inclined disk to become completely distorted;
Viscous force will smooth out the profile so the disk will have one outer inclined outer part, and one inner part flattened to the equatorial plane