A Brief Introduction to Linear MRI

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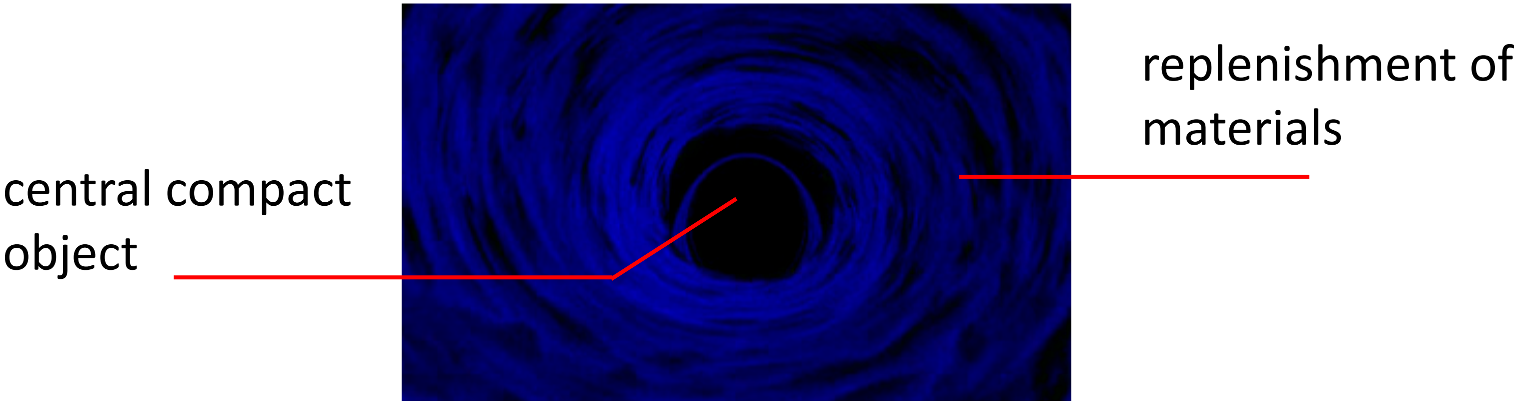
Accretion Disk and L Transport

Accretion disk: a structure formed by diffuse material in orbital motion around a massive central body.



What do we need?

- A theory on how viscosity causes orbiting material in the disk to spiral inward towards the central body, losing angular momentum

Accretion Disk and L Transport

General Navier-Stokes Equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{f} \tag{1}$$

Or

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla P + \eta \nabla^2 \boldsymbol{u} + \boldsymbol{f} \tag{2}$$

Because of the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{3}$$

Instability: Under perturbation based on the solution, an (uncontrollable) exponential evolution rather than small oscillations
(Jeans instability etc…)

Accretion Disk and L Transport

N-S equation and L transport in Accretion Disks

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot \boldsymbol{M} = -\rho \nabla \Phi \tag{19}$$

$$\boldsymbol{M} := \rho \boldsymbol{u}\boldsymbol{u} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} + (P + \frac{B^2}{8\pi})\boldsymbol{I} \tag{20}$$

The ϕ momentum equation of (20) gives AM conservation

$$\partial_t \left(\rho R v_\phi\right) + \frac{1}{R}\partial_R \left(R^2 M_{R\phi}\right) + \frac{1}{R}\partial_\phi \left(R M_{\phi\phi}\right) + \partial_z \left(R M_{z\phi}\right) = -\rho \partial_\phi \Phi \tag{22}$$

Integrate this over z and phi,

$$\frac{\partial \left(2\pi R \Sigma L\right)}{\partial t} + \frac{\partial}{\partial R}\left[2\pi R^2 \int_{-\infty}^{\infty} dz \left(\overline{\rho v_R v_\phi} - \frac{\overline{B_R B_\phi}}{4\pi}\right)\right] + 2\pi R^2 \left(\overline{\rho v_z v_\phi} - \frac{\overline{B_z B_\phi}}{4\pi}\right)\Bigg|_{-\infty}^{\infty} = 0 \tag{23}$$

With the over line denoting the azimuthal average and angular momentum

$$L = R v_K, \Sigma = \int \rho dz$$

Accretion Disk and L Transport

N-S equation and L transport in Accretion Disks

decompose the azimuthal and radial velocities into the stable mean and the fluctuations:

$$v_R = v_{R0} + \delta v_R, v_\phi = v_{\phi 0} + \delta v_\phi$$

$$\frac{\partial\left(2\pi R\Sigma L\right)}{\partial t}+\frac{\partial\left(\dot{M}_{\rm acc}L\right)}{\partial R}+\frac{\partial}{\partial R}\left[\boxed{2\pi R^2\int_{-\infty}^{\infty}dz\left(\overline{\rho\delta v_R\delta v_\phi}-\frac{\overline{B_RB_\phi}}{4\pi}\right)}\right]\boxed{+2\pi R^2\left(\overline{\rho\delta v_z\delta v_\phi}-\frac{\overline{B_zB_\phi}}{4\pi}\right)\Bigg|_{-\infty}^{\infty}} \quad (25)$$

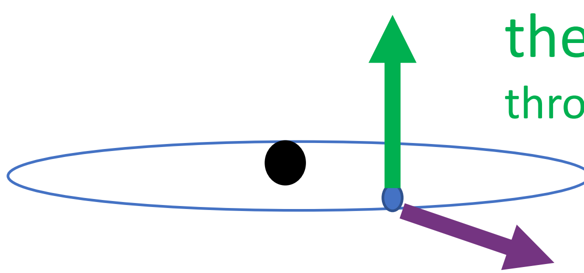
= 0

Angular momentum flux in the radial direction

Reynolds & Maxwell Stress

angular momentum flux in the vertical direction

through disk surface, by a disk wind



Accretion Disk and L Transport

N-S equation and L transport in Accretion Disks

$$\frac{\partial}{\partial R}\left[2\pi R^2\int_{-\infty}^{\infty} dz\left(\overline{\rho\delta v_R\delta v_\phi}-\frac{\overline{B_R B_\phi}}{4\pi}\right)\right].$$

Shear force: one component of the stress tensor $T_{r\Phi}$
Acts on a surface unit perpendicular to r and point towards Φ
Just like a *viscous force*!

α-hypothesis: $\overline{\rho\delta v_R\delta v_\phi}-\frac{\overline{B_R B_\phi}}{4\pi}\equiv\alpha P\approx\alpha\rho c_s^2$

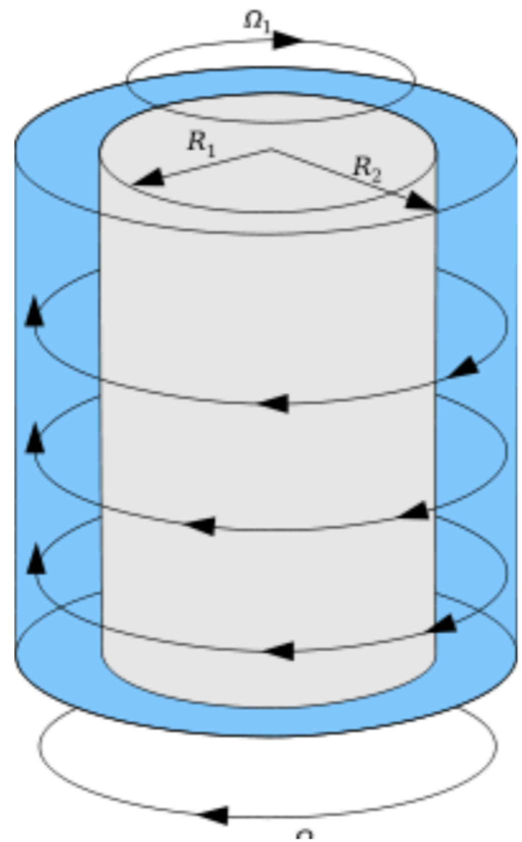
If there's no magnetic field,
we must have (pure hydrodynamics) turbulence!

Rayleigh Criterion and Background Knowledge

Taylor Couette Flow : a viscous fluid confined in the gap between two rotating cylinders

“Taylor-Couette flow has found particular applicability as a model for astrophysical accretion disks in determining the out-ward angular momentum flux…but have produced contradictory answers to these questions……Taylor-Couette experiments which do not match accretion disks”

–(F. Nordsiek et al.2014)



Rayleigh Criterion and Background Knowledge

Rayleigh Criterion:

Rayleigh's original paper (1880) ：

no viscosity, for an angular velocity distribution, the stability criterion:

$$\frac{d}{dr}(r^2\Omega)^2 > 0 \tag{5}$$

The angular momentum should increase outwards.

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P \tag{4}$$

Rayleigh Criterion and Background Knowledge

Rayleigh's Original Analysis

Rewriting (4) in the cylindrical co-ordinate system we have:

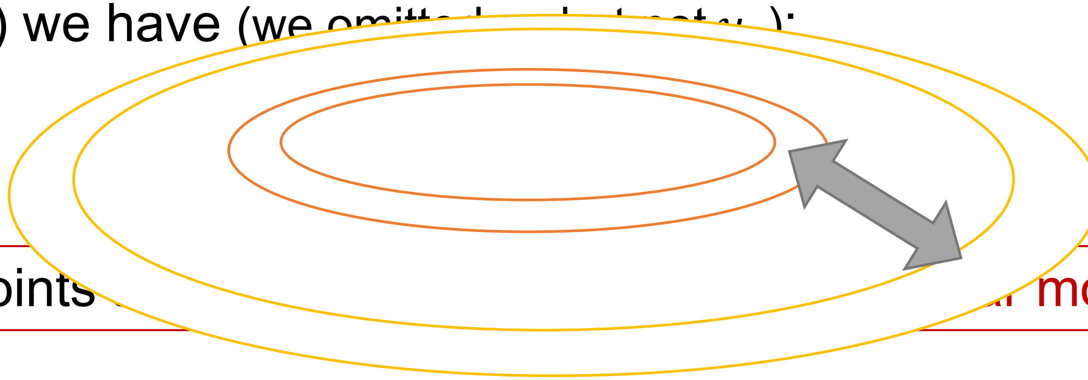
$$\begin{cases} \dfrac{\partial u_r}{\partial t} + (u_r \dfrac{\partial}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial}{\partial \theta} + u_z \dfrac{\partial}{\partial z}) u_r - \dfrac{u_\theta^2}{r} = -\dfrac{\partial}{\partial r}(\dfrac{P}{\rho}) \\ \dfrac{\partial u_\theta}{\partial t} + (u_r \dfrac{\partial}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial}{\partial \theta} + u_z \dfrac{\partial}{\partial z}) u_\theta + \dfrac{u_\theta u_r}{r} = -\dfrac{\partial}{r\partial \theta}(\dfrac{P}{\rho}) \\ \dfrac{\partial u_z}{\partial t} + (u_r \dfrac{\partial}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial}{\partial \theta} + u_z \dfrac{\partial}{\partial z}) u_z = -\dfrac{\partial}{\partial z}(\dfrac{P}{\rho}) \end{cases} \tag{6}$$

Stationary solution:

$$u_r \approx 0, u_z = 0, u_\theta = V(r), p(r) = \rho \int \frac{dr}{r} V^2 \tag{7}$$

Rayleigh Criterion and Background Knowledge

Combining the middle equation of (6) and the continuity equation (cylindrical) we have (we omitted ...):



$$2\pi r_1 dr_1 = 2\pi r_2 dr_2 \quad (8)$$

The flow points ... momentum.

Mind that this is NOT the ultimate criterion because this derivative is taken with respect to t.

"perturbation", swap two rings with the same amount of liquids (incompressible), keep their angular momentum unchanged, the increased energy prop to:

$$\{(\frac{L_2^2}{r_1^2}+\frac{L_1^2}{r_2^2})-(\frac{L_1^2}{r_1^2}+\frac{L_2^2}{r_2^2})\}dm=(L_2^2-L_1^2)(\frac{1}{r_1^2}-\frac{1}{r_2^2})dm \quad (9)$$

For the energy to be negative, angular momentum must increase with respect to r.

Rayleigh Criterion and Background Knowledge

Perturbation Analysis (Chandrasekhar 2003)

$$A(x,y,z,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y A_k(z)\exp\left[i\left(k_x x+k_y y+pt\right)\right]$$

Core idea: Emphasize one direction and decompose the rest
(Stability depends on p)

perturbation $0->u_r, V->V+u_\theta, 0->u_z, \frac{P}{\rho}->\frac{P}{\rho}+w$

$$u_r=u_r(r)e^{i(pt+m\theta+kz)}, u_\theta=u_\theta(r)e^{i(pt+m\theta+kz)}, u_z=u_z(r)e^{i(pt+m\theta+kz)}, w=w(r)e^{i(pt+m\theta+kz)}$$

$$\left\{ \frac{\partial u_r}{\partial t}+\frac{V}{r}\frac{\partial u_r}{\partial \theta}-\frac{V}{r}u_\theta=-\frac{\partial w}{\partial r} \right. \qquad (11)$$

Perturbation Analysis Results:
p is real as long as Rayleigh Criterion is satisfied!

Rayleigh Criterion and Background Knowledge

Viscosity?(B. Dubrulle et al 2011)

Constant Viscosity

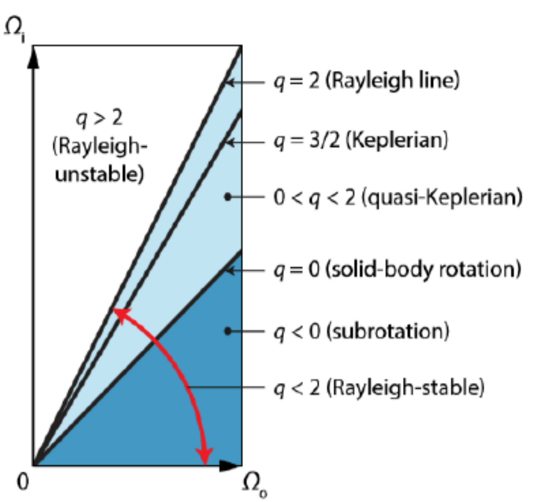
$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \boldsymbol{u} \tag{16}$$

There is also a stationary solution to that depending on the inner and outer boundaries:

$$u_z = 0, u_r = \frac{K}{r}, u_\theta = Ar^\alpha + \frac{B}{r^2} \tag{17}$$

Other Tries?(Numerical)

- **Different Viscosity**
- **Potential**
- **Higher Azimuthal Velocity/Re(Non-linear)**



Results:

No turbulence as long as Rayleigh Criterion is satisfied

Linear MRI

Magnetic Field

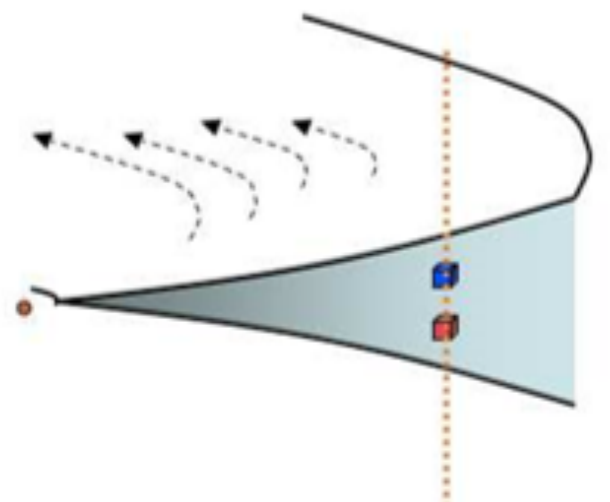
$$\frac{\partial}{\partial R}\left[2\pi R^2 \int_{-\infty}^{\infty} dz \left(\overline{\rho\delta v_R \delta v_\phi} - \frac{\overline{B_R B_\phi}}{4\pi}\right)\right].$$

Adding a magnetic field not only *might aid* the shear force Bur also gives turbulance!

The General Picture

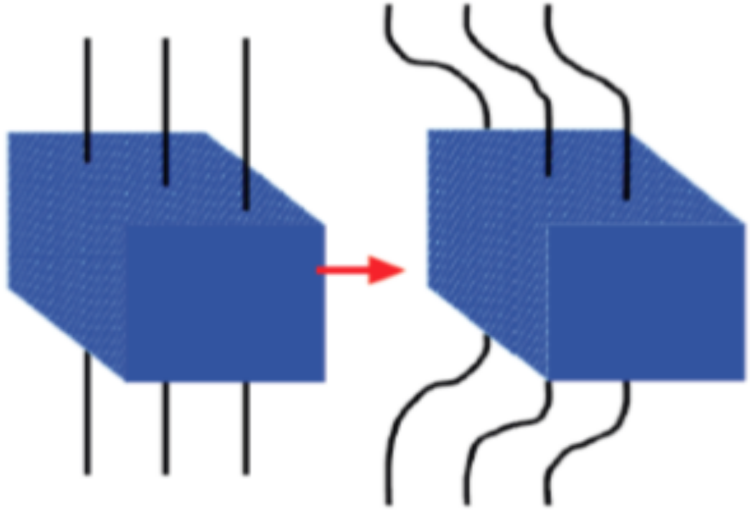
Flux Freezing: There’s a magnetic field with Z direction in a rotating disk, the field lines are also gyrating with the disk components.

Consider just two fluid points distributed vertically and the field line between them.



Linear MRI

Magnetic Field



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The General Picture

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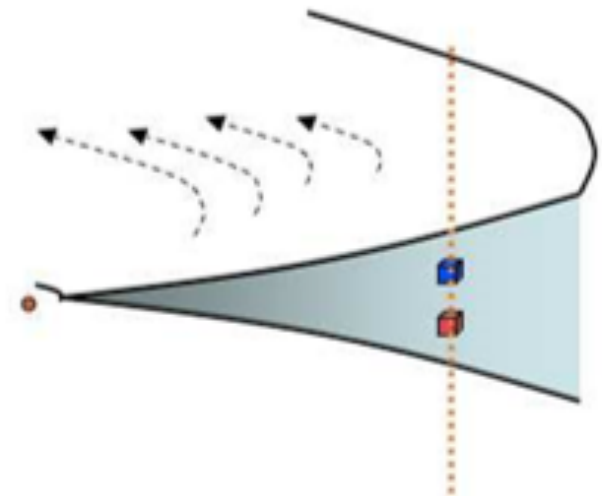
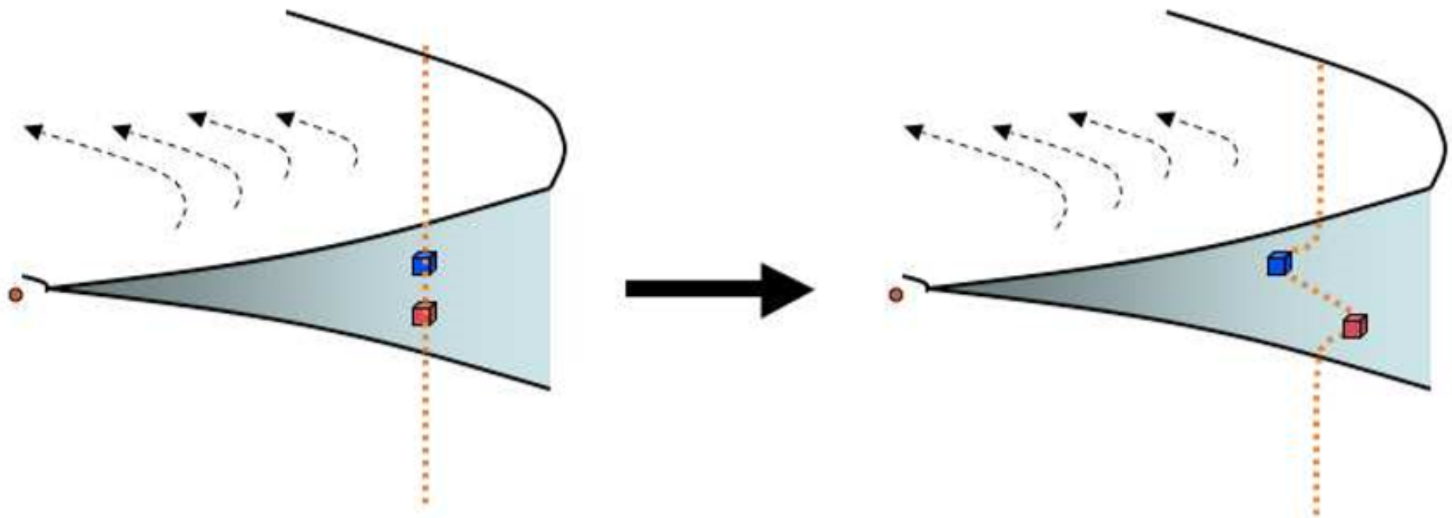


Consider just two fluid points distributed vertically and the field line between them.

Linear MRI

The General Picture

Azimuthal displacement->the field lines move along with them and will have an azimuthal component.



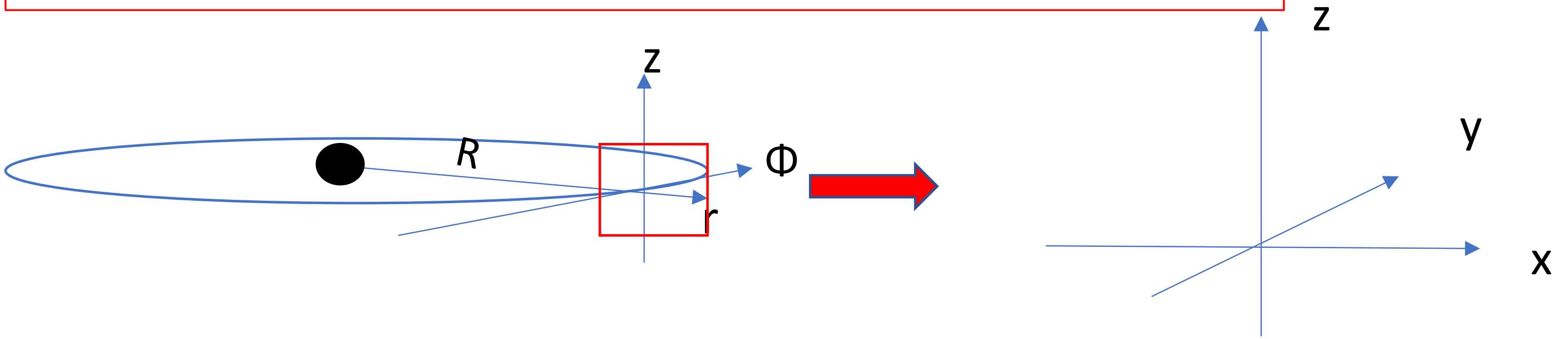
Tension of the field line will work to speed up one and slow down the other. So one will fall inwards and the other one outwards. (Enough turbulence that eventually break the field line attachments!)

Linear MRI

Derivation of MRI

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot \boldsymbol{M} = -\rho \nabla \Phi$$

Simplify with regional AND co-rotating co-ordinate system!



G acceleration $$-\frac{GM}{(R_0+x)^2}\boldsymbol{e}_x \approx -\frac{GM}{R_0^2}\left(1-2\frac{x}{R_0}\right)\boldsymbol{e}_x = \left(-\Omega_K^2 R_0 + 2\Omega_K^2 x\right)\boldsymbol{e}_x$$

Centrifugal $$\Omega_K^2\left(R_0+x\right)\boldsymbol{e}_x$$

Coriolis $$2\boldsymbol{v}\times\boldsymbol{\Omega_K}$$

Linear MRI

Derivation of MRI: Basic Equations

The original state before perturbation:

$$\boldsymbol{v_0} = -\frac{3}{2}\Omega x \boldsymbol{e_y}, (v_{0y} = -\frac{3}{2}\Omega x, v_{0z} = v_{0x} = 0), \boldsymbol{B_0} = B_0 \boldsymbol{e_z}, P = const, \rho = const \tag{30}$$

MHD equations(Continuity, momentum, induction. Refer to lec 01):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v_0}) = 0 \\ \dfrac{\partial \boldsymbol{v_0}}{\partial t} + (\boldsymbol{v_0} \cdot \nabla)\boldsymbol{v_0} = -\dfrac{\nabla P}{\rho} + \left[2\boldsymbol{v_0} \times \boldsymbol{\Omega_K} + 3\Omega_K^2 x \boldsymbol{e_x}\right] + \dfrac{1}{4\pi\rho}(\nabla \times \boldsymbol{B_0}) \times \boldsymbol{B_0} \\ \dfrac{\partial \boldsymbol{B_0}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B_0}) \end{cases} \tag{31}$$

Linear MRI

Derivation of MRI: Perturbation

First order terms

$$\begin{cases} \nabla \cdot (\boldsymbol{\delta v}) = 0 \\ \dfrac{\partial \boldsymbol{\delta v}}{\partial t} = -\dfrac{\nabla \delta P}{\rho_0} - \dfrac{1}{2}\Omega_K \delta v_x \boldsymbol{e_y} + 2\Omega_K \delta v_y \boldsymbol{e_x} + \dfrac{1}{4\pi\rho_0}(\nabla \times \boldsymbol{\delta B}) \times \boldsymbol{B_0} \\ \dfrac{\partial \boldsymbol{\delta B}}{\partial t} = \nabla \times (\boldsymbol{\delta v} \times \boldsymbol{B_0}) - \dfrac{3}{2}\Omega_K \delta B_x \boldsymbol{e_y} \end{cases} \tag{34$$

Uniform dependence:

$$\boldsymbol{\delta v} = \boldsymbol{\delta v} e^{\sigma t + ikz}, \boldsymbol{\delta B} = \boldsymbol{\delta B} e^{\sigma t + ikz}, \delta P = \delta P e^{\sigma t + ikz}$$

Non-dimensionalize:

Alven speed $v_A = \frac{B_0}{\sqrt{4\pi\rho}}$

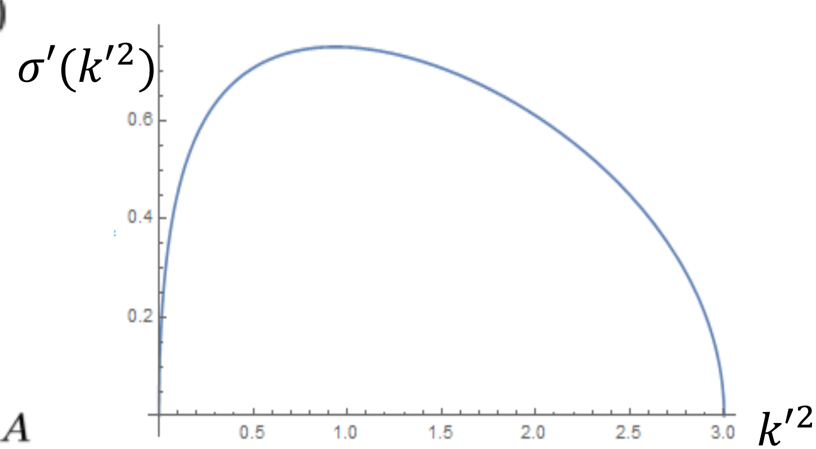
$$\sigma' \equiv \sigma/\Omega_K, \quad v' \equiv v/v_A, \quad B' \equiv B/B_0, \quad k' \equiv k v_A/\Omega_K$$

Linear MRI

Derivation of MRI: Results

$$\sigma'^4 + \sigma'^2\left(2k'^2+1\right) + k'^2\left(k'^2-3\right) = 0$$

Real roots when $0 < k' < \sqrt{3}$



Restoring dimensions:

- Critical Wavenumber: $k_c = \sqrt{3}\Omega_K / v_A$
- Critical Wavelength: $\lambda_c = \dfrac{2\pi}{k_c}$

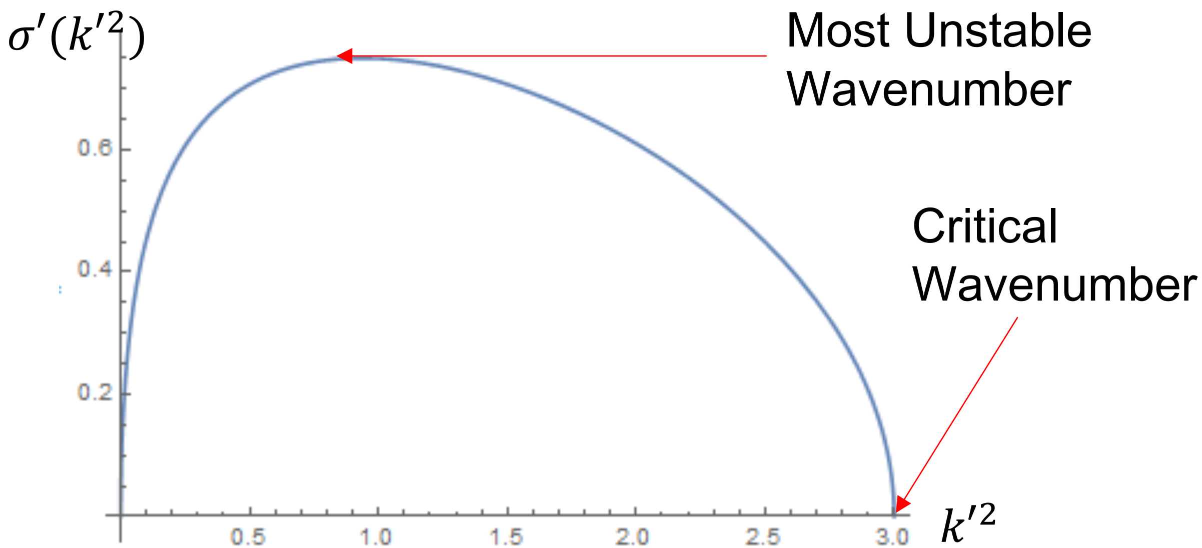
How big must the scale of perturbation be (depends on a certain magnetic strength!)

- Most Unstable Wavenumber: $k_m = \sqrt{15/16}\Omega_K / v_A$

$$\sigma^{\max} = (3/4)\Omega_K$$

Linear MRI

Derivation of MRI: Results



Why large in the middle ?

- Big k': Field lines restore too easily
- Small k': Field lines will not restore completely, but takes forever to influence velocity(exchange angular momentum)

Asymptote paradox

$\sigma^{\max} = (3/4)\Omega_K$ for no matter how small a B/v_A?

Linear MRI

Generalization:

More generally for other potentials we have

$$\sigma^4 + \sigma^2 \left[\kappa^2 + 2\,(kv_A)^2\right] + (kv_A)^2 \left[(kv_A)^2 + \frac{d\Omega^2}{d\ln R}\right] = 0 \qquad (42)$$

$$\kappa^2 \equiv 4\Omega^2 + d\Omega^2/d\ln R = d(R^2\Omega)^2/d\ln R$$

The condition for this region to exist is the MRI criterion:

$$\boxed{\frac{d\Omega}{dr} < 0} \qquad (43)$$

Could apply to other models than Accretion disks!