The Cooling of White Dwarves

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ABSTRACT: We will start from an inaccurate model where we approximated the cooling of white dwarves with blackbody radiation. Then, after discussions concerning opacity, degenerate pressure and stuff, I modified the radiation luminosity and derived another cooling curve.

1 The Cooling of White Dwarves——Blackbody

1.1 Blackbody Description

The cooling time of a white dwarf, is the time for a white dwarf of known initial mass, temperature and luminosity to cool to another certain temperature. in some elementary astrophysics courses we will see this kind of approximation in solving such a problem: Treat the WD as a blackbody, the composition's molecular weight is C (E.g. for oxygen C=16，Carbon C=12), the total energy of WD is the internal energy of all atoms, and we get a differential equation.

For the blackbody radiation：

$$p = \frac{\alpha}{3}T^4, M = \frac{1}{4}c\alpha T^4 = \sigma T^4 \tag{1}$$

Where $\sigma = 5.67 \times 10^{-5}$(c.g.s units)

1.2 Solving for cooling time

In this case we need the initial radius R and mass M，set C=12 for a carbon WD。The total number of atoms is $N = \frac{M}{12m_p}$, therefore the internal energy is:

$$E = \frac{3MkT}{24m_p} \tag{2}$$

Luminosity would be

$$L = 4\pi R^2\sigma T^4 \tag{3}$$

Therefore we have differential eqn:

$$\frac{3MkdT}{24m_p dt} = -4\pi R^2 \sigma T^4 \tag{4}$$

The solution is (usually the third power of the final temperature is much, much smaller than the initial)

$$\frac{1}{T^3} - \frac{1}{T_0^3} \approx \frac{1}{T^3} = \frac{12\pi R^2 \sigma \times 24 m_p}{3Mk} t = \frac{32\pi R^2 \sigma m_p}{Mk} t \tag{5}$$

Therefore, the time needed to cool down to L is

$$\tau(L) = \frac{Mk}{32\pi R^2 \sigma m_p} (\frac{4\pi R^2 \sigma}{L})^{3/4} \tag{6}$$

2 Discussion

What might be wrong with this model?

Well first it doesn't consider the relation between M and R. We will see later than for a WD $M^{1/3}R = const$. Therefore we only need one of those parameters.

Another problem that is easy to think about is, does the equations of state for radiation field apply? We know that the interior of the WD is actually degenerate, isn't that contradictory?

However, a fact that could to some extent save this assumption is that, from observation we know that the surface temperature is on the order of $10^4 K$, and the density is smaller than $10^2 g/cm^2$, so we can reasonably approximate the WD to consist of two layers: a degenerate core and a thin blackbody radiative surface. Even in the degenerate core the CARBONS are not degenerate, it's only the electrons.

The last question is, is the form of total internal energy $\frac{3}{2}NkT$? The energy of the WD? This can also be justified after we find that although the *pressure* of the degenerate electron gas contributes to the bulk of the stellar pressure, the fermi energy of the electrons is still much smaller than the internal energy of the carbon atoms.

So after these discussions, let's formally derive a modified version of the cooling time.

3 Cooling of white dwarves ——Modified

3.1 Basic equations for Stellar structure

The physical structure of stars is quite complicated, but still most of the times under the assumption that 1)stars are spherically symmetric (2) Stars are stable and maintain hydrostatic equilibrium, we can use some basic equations to describe it.

1.Mass distribution

For every differential shell in the sphere we have

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{7}$$

Where M(r) is the total mass included within radius r, and $M(r) = \int_0^r 4\pi r^2 \rho(r) dr$.

2.Hydrostatic Equilibrium

The gravity for every cell maintains an equilibrium with pressure (radiative pressure, idea gas, degenerate pressure, any kind):

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \tag{8}$$

3.Energy equilibrium

ϵ is the energy generated by unit mass per time for the star and L is the energy radiated outwards per unit time, still for every layer of shell structure we have:

$$\frac{dL}{dM} = \epsilon \tag{9}$$

For the WD, from the first law we have (for unit mass)

$$dQ = du - \frac{P}{\rho^2}d\rho = \frac{\partial u}{\partial T}dT + (\frac{\partial u}{\partial \rho} - \frac{P}{\rho^2})d\rho \tag{10}$$

Since $dS = \frac{dQ}{T}$ is a total differential， we have

$$\frac{\partial u}{\partial \rho} = \frac{P}{\rho^2} - \frac{T}{\rho^2}(\frac{\partial P}{\partial T})_\rho \tag{11}$$

Actually this can also be obtained from Maxwell equations, write U as a function of T and V：

$$du = T(\frac{\partial S}{\partial T})_V dT + [T(\frac{\partial P}{\partial T})_V - P]dV \tag{12}$$

For $c_\rho = c_V$, the fix volume capacity per unit mass, immediately we have

$$du = T(\frac{\partial S}{\partial T})_\rho dT - \frac{1}{\rho^2}[T(\frac{\partial P}{\partial T})_\rho - P]d\rho = c_\rho dT - \frac{1}{\rho^2}[T(\frac{\partial P}{\partial T})_\rho - P]d\rho \tag{13}$$

Substitute back into Eqn (10) to get

$$dQ = c_\rho dT - \frac{T}{\rho^2}\frac{\partial P}{\partial T}d\rho \tag{14}$$

Taking derivate with respect to time, we have:

$$\epsilon_g = \frac{T}{\rho^2}\frac{\partial P}{\partial T}\dot{\rho} - c_\rho \dot{T} \tag{15}$$

Therefore for the WD with no other means of generating heat such as nuclear fusion and nearly-unchanging total density, we have from eqn (9):

$$L = -\int_0^M c_\rho \dot{T} dm \tag{16}$$

Where T is the non-degenerate surface temperature and approximately the characteristic temperature of whole star.

4.The propagation of energy

∇, the temperature gradient is defined as

$$\nabla = \frac{dlnT}{dlnP} = \frac{PdT}{TdP} \tag{17}$$

Combining (7) and (8), we have

$$\frac{dT}{dM} = -\frac{GMT}{4\pi r^4 P}\nabla \tag{18}$$

The specific form of the density gradient in a **Radiative Diffusion** dominated regime is

$$\nabla = \frac{3}{64\pi\sigma G}\frac{\bar{\kappa}LP}{MT^4} \tag{19}$$

Where κ is opacity, σ is the SB constant. For typical free-free absorption process, a mean approximated opacity can be expressed as:

$$\bar{\kappa} = \kappa_0 \rho T^{-3.5} \tag{20}$$

3.2 Degenerate Pressure

We will give a simple derivation of the fore-mentioned M-R relation of WD. For hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho\frac{GM}{r^2} \tag{21}$$

For Fermi pressure, we have discussed in class that

$$P \propto n_e^{5/3} \propto \rho^{5/3} \tag{22}$$

So there comes the scaling relation

$$\frac{P}{R} \propto \frac{M^{5/3}}{R^6} \tag{23}$$

The RHS of (21) can be written as

$$-\rho\frac{GM}{r^2} \propto -\frac{M^2}{R^5} \tag{24}$$

Therefore

$$M^{1/3}R \approx Const. \tag{25}$$

But since what we want is the surface, why do we need the interior anyway? In fact, this is actually for calculating the very boundary of the interior and the near-ideal surface by equating the two pressures:

$$\frac{k}{(C=12)m_p}\rho T = \frac{h^2}{20m_e m_P}(\frac{3}{\pi m_p})^{3/2}(\frac{\rho}{\mu_e})^{5/3} = P \tag{26}$$

In fact the entire WD is going to be around this fixed density and temperature at one specific time.

We have previously derived that

$$L = -\int_0^M c_\rho \dot{T} dm = -c_\rho M \frac{dT}{dt} \tag{27}$$

And justified that

$$c_\rho = c_v = (\frac{\partial u}{\partial T})_V \approx \frac{3}{2}\frac{k\rho}{Cm_p} \tag{28}$$

Again, arrange the P and T gradient

$$\begin{cases} \dfrac{dP}{dr} = -\dfrac{GM\rho}{r^2} \\ \dfrac{dT}{dr} = -\dfrac{3\bar{\kappa}\rho}{64\pi\sigma}\dfrac{L}{r^2T^3} \end{cases} \tag{29}$$

and bring in the specific form of opacity (20) to get the explicit form of equation (18):

$$P = (\frac{64\pi\sigma GM}{12.75\kappa_0 L})^{1/2} T^{4.25} \tag{30}$$

Combining (26) and (30), effectively three equations with three variables (P,ρ, T), we finally get the temperature

$$T^{3.5} = (\frac{C_1}{C_2})^2 = B\frac{L}{M}, B = \frac{2.82\times 10^{-7}\kappa_0 k^2 \mu_e^5}{C^5 m_p^2 64\pi\sigma G} \tag{31}$$

3.3 Cooling time

Therefore, our differential equation should be changed to

$$\frac{dT}{dt} = -\frac{2}{3}\frac{Cm_p}{\rho k B}T^{3.5} \tag{32}$$

$$\frac{1}{T^{2.5}} \approx \frac{5}{3}\frac{Cm_p}{\rho k B}t \tag{33}$$

To cool to a specific luminosity, we need

$$\tau(L) = \frac{2}{5}B^{2/7}\frac{3}{2}\frac{\rho k}{Cm_p}(\frac{M}{L})^{5/7} \tag{34}$$

This will be our result of the modified cooling timescale.

If we substitude R with

$$\rho \propto \frac{M}{R^3} \propto M^2 \tag{35}$$

Then

$$\tau \propto \frac{M^{19/7}}{L^{5/7}} \tag{36}$$

This is quite different from the original form of (6).