Dust Diffusion in Protostellar Disks and its Implications on Super-Earth Formation

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1 Introduction

1.1 Planet Formation and *In Situ* Accretion

A protostellar disk is the embryo of a solar system, consisting of 99% gas(H, He) and 1% dust, which will dissipate and evolve into numerous kinds of planets, including terrestrial planets, gas giants, ice giants, and super-Earths, in a typical timescale of 1-10 Myrs. Super-Earths are rocky planets with typical radii of 1-4 $R_\oplus$, masses of $10M_\oplus$ at distances of 0.05-0.3AU or beyond, commonly discovered in 20-30% of solar systems (Howard et al. 2010; Batalha et al. 2013; Petigura et al. 2013; Dong & Zhu 2013; Fressin et al. 2013; Rowe et al. 2014), and a difficult problem for the classical planet formation theory to handle.

A *Classical* planet formation process starts with planetesimals accumulating into a solid core, and later steady accretion of more planetesimals and gas onto the core, forming a growing envelope around it. Given enough time, the gas envelope might grow large enough, or to be exact the Gas-to-Core mass Ratio (GCR) grow to such that it could no longer maintain hydrostatic equilibrium, runaway accretion kicks in and the planet explode into gas giants (Perri & Cameron 1974; Harris 1978; Mizuno et al. 1978; Mizuno 1980; Stevenson 1982). The **"Critical Mass"** refers to a core mass that for which the gas envelope around it would grow large enough and runaway accretion occurs just within the gas disk lifetime, and it's around the typical value of 10 $M_\oplus$.(Stevenson 1982)

However, many super-Earths or close-in super-Earths indeed have core masses that exceed this common limit yet they did not explode into Jupiters within the disk lifetime. Some plausible explanations include that super earths may actually have fast accretion rates, but once the parent nebula dissipates, atmospheric loss driven by the sun's stellar irradiation would be able to drive away gas(e.g., Lopez & Fortney 2013; Owen & Wu 2013) . But this is a temporary effect after gas dissipation, and most gas giants could not be blown into a dwarf super-earth in a small period, so there might be more secular influences on the original accretion speed. Similar problems exist with migration (Kley & Nelson 2012), because what halts or determines the direction of migration remains unclear, so it's still very important that we consider the forming of super-Earths by *in situ* accretion, though there must be other explanations as to how it avoided runaway accretion. The simplest thought is to focus on the steady accretion of gas onto the core prior to runaway. If it's slowed down abnormally, then this rocky core might not be able to grow to the GRC that triggers the instability in the limited life span of the protostellar accretion disk. To put it in another way, a hindered accretion rate would result in a Critical Mass higher than normal.

Lee et al. (2015) have simulated the gas accretion process and found that 10 $M_\oplus$ cores would inevitably reach the instability and grow into gas giants in normal circumstances, so they proposed that super-Earth cores may mostly have formed towards the very end of the protoplanetary disk life span or that the dust density is too high. The second possibility is consistent with a hindered accretion rate (we will refer to this again in section 3).

1.2 Pebble Accretion and Pebble Isolation Mass

Different from the classical model, recent studies have proposed a modified paradigm of pebble accretion, in which pebbles - dust particles of characteristic length 0.1-10cm - can also be accreted onto the core along with the gas (Johansen & Lacerda 2010; Ormel & Klahr 2010; Lambrechts & Johansen 2012; Morbidelli & Nesvorny 2012) after the solid core is formed. This model gives a typical critical mass of 100 M_E because the pebbles can help support hydrostatic equilibrium. However, when the solid core reaches the **"Pebble Isolation Mass"** , it could generate a pressure bump that traps drifting pebbles outside its orbit (Rice 2006; Bitsch et al 2018). That's when accretion of solids abruptly ceases, the envelope loses the support and the critical mass suddenly drops to a value usually smaller than the pebble isolation mass itself so the planet immediately undergoes runaway accretion, so pebble isolation mass more or less becomes the new criterion for runaway accretion (Lambrechts et al. 2014).

Lambrechts et al. (2014) used hydrodynamical simulations to infer a pebble-isolation mass typically $10 - 20M_E$ given by:

$$M_{\rm iso} \approx 20 \left(\frac{H/R}{0.05}\right)^3 \mathrm{M}_\oplus \tag{1}$$

Basically, ignoring the viscous effects, we understand that this criterion increases with the radius of the planet's orbit. Therefore, the rocky cores distant from the sun needs to be larger that those near the sun to be able to undergo runaway accretion. This is a potential mechanism (maybe more plausible that the standard model) to explain the dichotomy between different planets, with ice giants never reaching the pebble isolation mass required for their orbital radius while the gas giants can in a short while.(Venturini & Helled 2017; Frehlik & Murray-Clay 2017)

In spite of the differences, the two processes are similar in many aspects, both involving a solid core steadily accreting gas (or gas and grains) until it reaches a criterion of 10-20 $M_\oplus$, which means the dichotomy between super-Earths and gas giants could not be explained by the change of runaway criterion alone. This paper would start from reviewing and simulating the pebble isolation process from the perspective of the diffusion of dust (Section 2), and ends by thoroughly discussing how the results might shed some new light upon formation of super-Earths and gas giants apart from the change of runaway criterion already emphasized.

2 Dust Diffusion and Pebble Isolation

2.1 The Dust Barrier in Transition Disks

Although dust only takes up 1% of disc mass, it contributes bulk of the opacity and emission lines(D' Alessio et al. 1999, 2001). And what we observed in transition disks, is that the dust profile seems to be gapped or even truncated. Some of them appears to be entirely devoid of circumstellar material within certain of the star. This phenomenon can be explained through pebble isolation (Kenyon & Hartmann 1995; Ercolano et al. 2011; Koepferl et al. 2013; Espaillat et al. 2014).

The physical picture is as thus: in the steady gaseous disc, the azimuthal velocity of the gas is not strictly Keplerian, it yields to the momentum equation:

$$\frac{v_\phi^2}{R} = \frac{V_K^2}{R} + \frac{1}{\rho}\frac{dP}{dR} \tag{2}$$

Where V_K is the Keplerian velocity.

In most places where $\frac{dP}{dR}$ is negative, the velocity is sub-Keplerian. When the dust particles are dragged backwards by the gas, they lose angular momentum and spirals inward. However, a growing planet could open a partial gap around its orbit in the gas profile. The gas would be pushed to two sides (mainly outside), and a pressure maximum would be generated out of its orbit. In the region of the disc where $\frac{dP}{dR}$ is positive, the gas velocity will become super-Keplerian, and the drag force should cause dust particles to speed up and move outwards towards the pressure maxima where it reaches equilibrium and accumulates (Rice 2006; Bitsch et al 2018).

The simulation of this effect could be done with many methods, but in this paper, we approach this problem by applying the dust diffusion equation given by Clarke & Pringle (1988). By using this method, we can make reasonable approximations and go through all the important elements with very clear physical pictures, and obtain a relatively credible figure consistent with other simulations, but with much simpler numerical calculation.

It's very easy to simulate in 2-D manner the gap in the gas density created by a planet in FARGO3D (Benitez-Llambay & Masset 2016). The software uses a default unit system, where central star mass=1, orbital radius=1 and gravity constant G=1. The initial gaseous disk parameters with $\frac{H}{R} = h = h_0 R^\beta, \Sigma = \Sigma_0 R^{-\alpha}$, 100 mesh from r=0.5-2.5 in the radial direction, parameter as follows:

- h_0: 0.05
- Σ_0: 6.3661977237e-4
- ν_0: 1.0e-5
- α: 1.0
- β: 0.25

According to hydrodynamic equations, for a Keplerian accretion disc with steady accretion rate M, away from the inner edge we have (Pringle 1981)

$$u_{\rm g} = -\frac{3\nu}{2R} = -\frac{\dot{M}}{2\pi\Sigma_0 R} \tag{3}$$

So in this case, according to the α hypothesis (Shakura & Sunyaev 1972) the steady radial velocity would be:

$$u_{\rm g} = -\frac{3\nu_0 R}{2R} = -\frac{3}{2}\nu_0 \tag{4}$$

So it would be a constant for a large range and sensitive to perturbations. Also, we have $P = c_s^2\Sigma = (H\Omega)^2\Sigma$. This reminds us that to create a pressure maxima, we not only need the gas density to increase over r in a region, but the increasing effect needs to be enough to cancel out the reduction of sound speed c_s.

We can put in planets of different sizes and let the codes run for 5000 time units. We could see from (FIGURE 1) that they create different gaps in the gas profile along with density waves of different magnitude. The **Super-earth sized** planet of 0.0001 solar mass (middle) should be the typical planet core we are interested in, so we will be using this data in further steps.

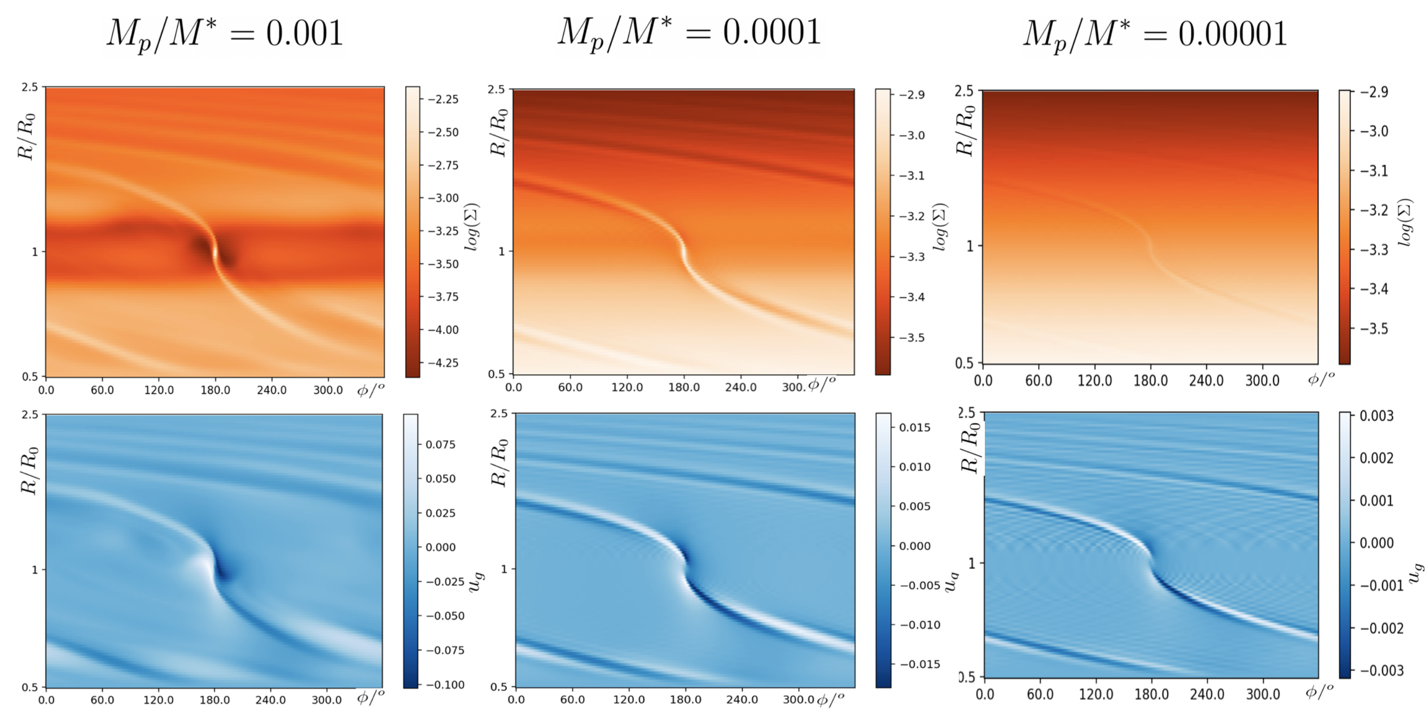

Figure 1: The effect on gas profile created by planets of different mass, representing Jupiter, super-Earth, and Earth. Snapshot taken after 5000 timescale units. Orange profile denotes the gas density and blue profile indicates the gas radial velocity (outwards as positive direction)

2.2 Numerically Solving Diffusion Equation

In an axisymmetric flat gaseous accretion disk, we can predict movements of trace contaminants (in our case, dust) by applying the diffusion equation (Clarke & Pringle 1988):

$$\Sigma(R)\frac{\partial C(R,t)}{\partial t} + \Sigma(R)u_{\mathrm{g}}(R)\frac{\partial C(R,t)}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\zeta\nu(R)\frac{\partial C(R,t)}{\partial R}\right) \tag{5}$$

Where Σ, σ is the gas and dust/particle/contaminant density, $C = \frac{\sigma}{\Sigma}$ is the concentration or dust-to-gas ratio, u_g is the radial velocity of the gas, ζ is the ratio of diffusion coefficient over effective viscosity and should roughly be a constant throughout the disk smaller than 1.

This equation is based on the assumptions that the small amount of dust would quickly reach the same radial velocity as the gas, and when we also assume that the feedbacks of dust grains on gas is negligible so the u_g and Σ, once come into shape, will not change in time with the evolution of dust.

In our case, the u_g and Σ profile is that of the perturbed state in (FIGURE 1, comparing with that of no planet perturbation). We extract the 1-D data from this state by taking azimuthal mean (FIGURE 2) to substitute into the equation, then we use Jacobi Iteration (see Appendix) to solve this ODE with open boundary condition $\partial_R C|_{bound} = 0$ and initial condition where $\sigma = 0.0001$ everywhere, which gives the results in (FIGURE 3). This clearly shows the inward migration of dust being blocked.

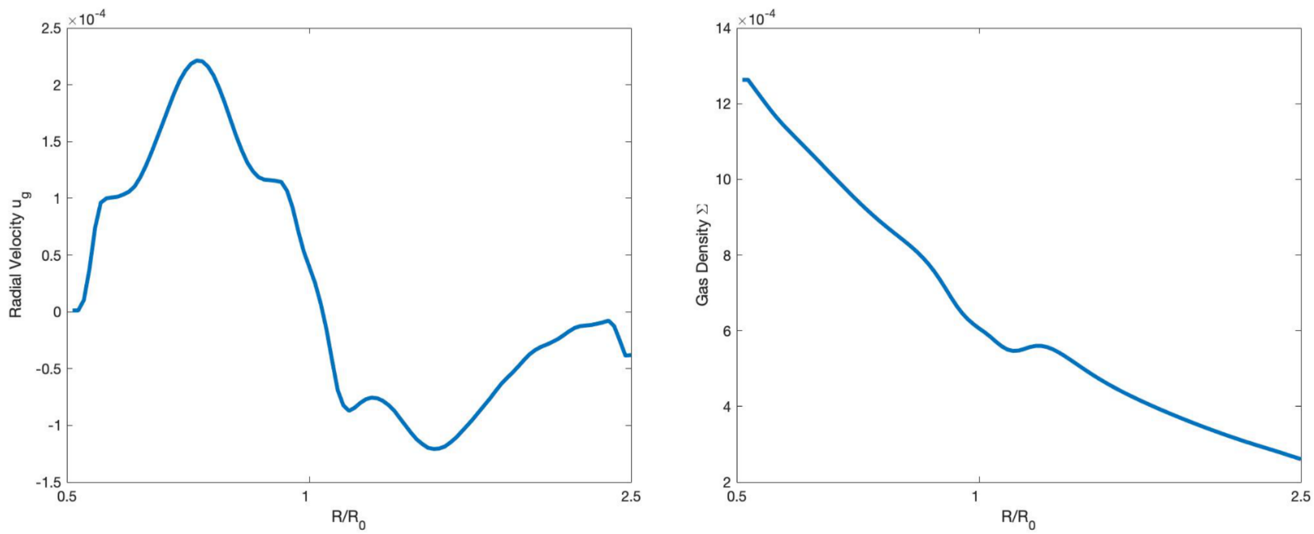

Figure 2: The (azimuthal) mean radial velocity and gas density in FARGO default unit system extracted from the 2-D data

2.3 Gas Drag

As we fore-mentioned, equation (5) does not take into account the effect of gas drag on dust grains, but assumes that dust particles have all reached the speed of the gas at the very beginning. From this result, we can see that although the early studies attributed pebble isolation to pressure gradient and hydrodynamical gas drag, diffusivity itself is also capable of creating this effect, with these two mechanisms being mutually inclusive. In our setup, the pressure gradient and the gas density gradient is very similar, so when σ is roughly a constant, the maxima of Σ will be the minimal of concentration C, so dust will accumulate towards the density sigma under the influence of diffusivity.

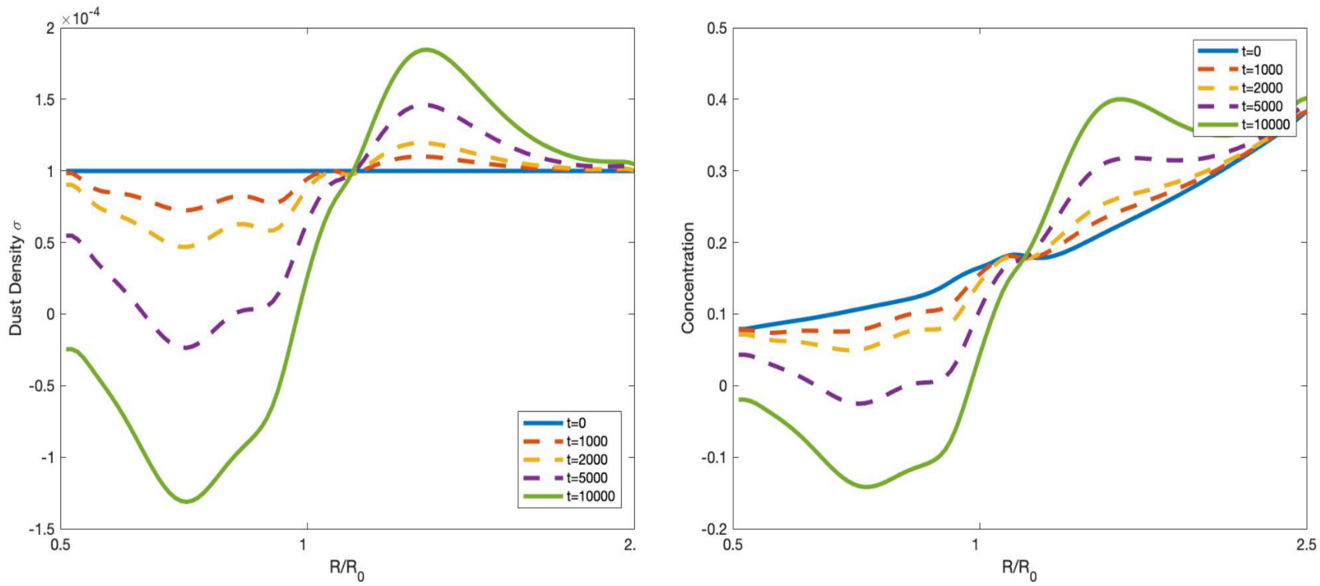

Figure 3: The evolution of dust density (left) and dust-to-gas ratio (right) calculated by our numerical methods. Snapshots at different time are plotted out together (FARGO default unit). This gives us a good picture of how the inward migration of certain dust particles is blocked. ζ_0 is assumed to be 0.1, we will give a clear reason why we chose this number in section 2.3

If we want to see the different effect on dust of different sizes (pebbles, grains etc), we can try to include the gas drag. The contaminant diffusion equation only represents an ideal state where dust reaches the very same radial velocity as the gas. In fact, according to azimuthal components of hydrodynamical momentum equation, the complete gas-dust interaction should yield to:

$$\begin{cases} -2v_p\Omega_k = \dfrac{u_g - u_p}{\tau_s} \\ \dfrac{1}{2}u_p\Omega_k = \dfrac{v_g - v_p}{\tau_s} \\ -2v_g\Omega_k = -\dfrac{\rho_p}{\rho}\dfrac{u_g - u_p}{\tau_s} + 2\eta\Omega_k \\ \dfrac{1}{2}u_g\Omega_k = -\dfrac{\rho_p}{\rho}\dfrac{v_g - v_p}{\tau_s} + \dfrac{1}{2}\xi\Omega_K \end{cases} \tag{6}$$

where v denotes the azimuthal velocity relative to Keplerian, τ_s is the stopping time and

$$\eta = -\frac{h^2\Omega_k R}{2}\frac{\partial \ln P}{\partial \ln R} \tag{7}$$

$$\xi = \frac{2\partial}{R^2\Omega_K\partial R}\left(\Sigma\nu R^3\frac{\partial\Omega_k}{\partial r}\right) \tag{8}$$

On a small timescale, while still neglecting the feedbacks, we can just obtain from the first 2 equations and a given u_g and $v_g$that:

$$u_p = \frac{u_g + 2v_g\Omega_k\tau_s}{1 + (\Omega_k\tau_s)^2} = \frac{u_g + 2v_g S_t}{1 + S_t^2} \tag{9}$$

We can see from the data of FIGURE 4 that the azimuthal velocity is not negligible compared to the radial velocity.

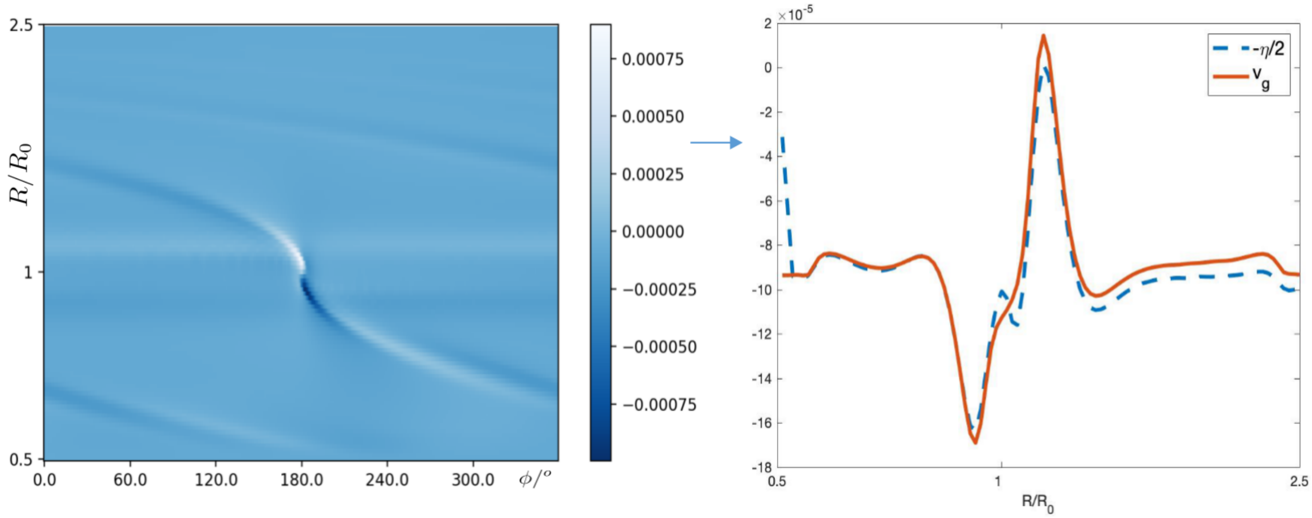

Figure 4: The 2-D relative azimuthal velocity v_g profile calculated from FARGO3D simulation data (left) and the azimuthal mean of which compared to $-\frac{1}{2}\eta$ (right) over radial direction.

This is normal because according to equation 2 we have:

$$v_g = \Omega_k R \left(\sqrt{1 + \frac{h^2}{2}(\frac{\partial \ln P}{\partial \ln R})} - 1 \right) \approx -\frac{1}{2}\eta \tag{10}$$

So azimuthal velocity and η should be on the same order.

The true radial velocity of the dust (to replace u_g in equation 5) should be related to the Stokes number, in this case written by Takeuchi & Lin (2005) as

$$S_t = \frac{\pi s_{\mathrm{p}} \rho_p}{2\Sigma} \tag{11}$$

where s_p is the size of particle and ρ_p is the intrinsic density. The Stokes number has a typical value of 1 at 1AU for normal grain particles. In our case, if we consider super-Earths of distance 0.1-0.3 AU, then the typical Stokes number would be on the order of 0.1. If we assume our dust particles have such an intrinsic density that for 1cm objects $S_t = 0.1$ at orbital distance 1 (distance of the planet), then the Stokes number of this particle would be a function of R. But in this case, we will first neglect the dependence of S_t on radius and assume $S_t = (\frac{s_d}{10cm})$, and since the diffusion coefficient ζ, on the contrary, decrease with the size of the particles (Youdin & Lithwick 2007), we will simply assume that $\zeta_0 = \frac{0.1cm}{s_d}$.

For different sized particles with different stokes numbers and diffusion coefficient, the gapping effect would be different as shown in FIGURE 5. the bigger dust grains (pebbles) are more likely to be totally blocked, a result consistent with the pebble isolation effects simulated in other studies (e.g. Owen 2014).

2.4 Feedback and The Flattening Effect

In the previous sections, we have assumed that the feedback of dust on gas is negligible so u_g and Σ do not change with time. But even if the original dust density is small, the bump in dust density

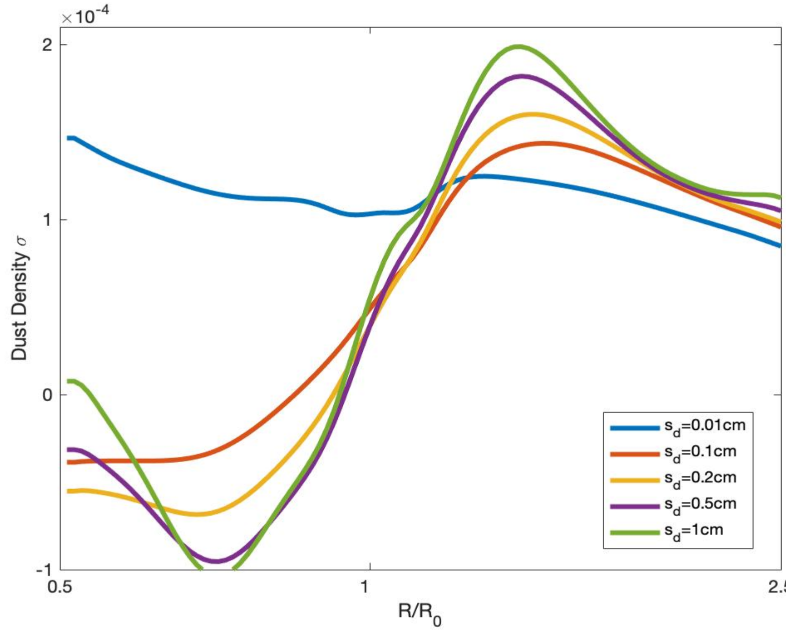

Figure 5: The dust density profile $\sigma(R)$ after 10^4 timescale for particles of different size, with same uniform initial distribution

outside the barrier would be piled up to a large amount, and on a longer term it will inevitably lead to feedback to a certain extent.

We can derive this from equation set 6, defining the velocity difference:

$$\Delta v := v_g - v_p, \quad \Delta u := u_g - u_p \tag{12}$$

From equation set 6 we get:

$$\begin{cases} -\Delta v = \eta - \frac{1}{2}(1+C)\frac{\Delta u}{S_t} \\ \Delta u = \xi - 2(1+C)\frac{\Delta v}{S_t} \end{cases} \tag{13}$$

Considering the drag effect, when particles are accumulating ($u_p = 0$) in regions where $\eta < 0$ and gas azimuthal velocity is super-Keplerian, C would grow until particles and gas reach the same azimuthal velocity:

$$-\Delta v = \eta - \frac{1}{2}(1+C)\frac{\Delta u}{S_t} = 0 \tag{14}$$

And we obtain a critical concentration:

$$C_{crit} = \frac{2\eta S_t}{u_g} - 1 \tag{15}$$

This concentration corresponds to a zero pebble velocity: after reaching this concentration, the dust particles will "flatten out" to accumulate somewhere farther away from the star because more contribution to the concentration will give an outward radial velocity and would be self-contradictory.

We should note that when feedbacks are non-negligible, the gas parameters u_g and Σ will both change with time. In this case, we can assume the relative change in the sensitive u_g is much quicker than that of Σ (or η) and v_g, so when u_p is zero, we can obtain the "new" u_g as well as the previous concentration from the first **three** equations of set (6). The new u_g should be:

$$u_g = -2v_p S_t = -2v_g S_t \tag{16}$$

When we are considering this in the diffusion equation, we can simply let u_p gradually reduce to zero on an appropriate timescale to generate a zeroth order approximation of the "flattening" effect.

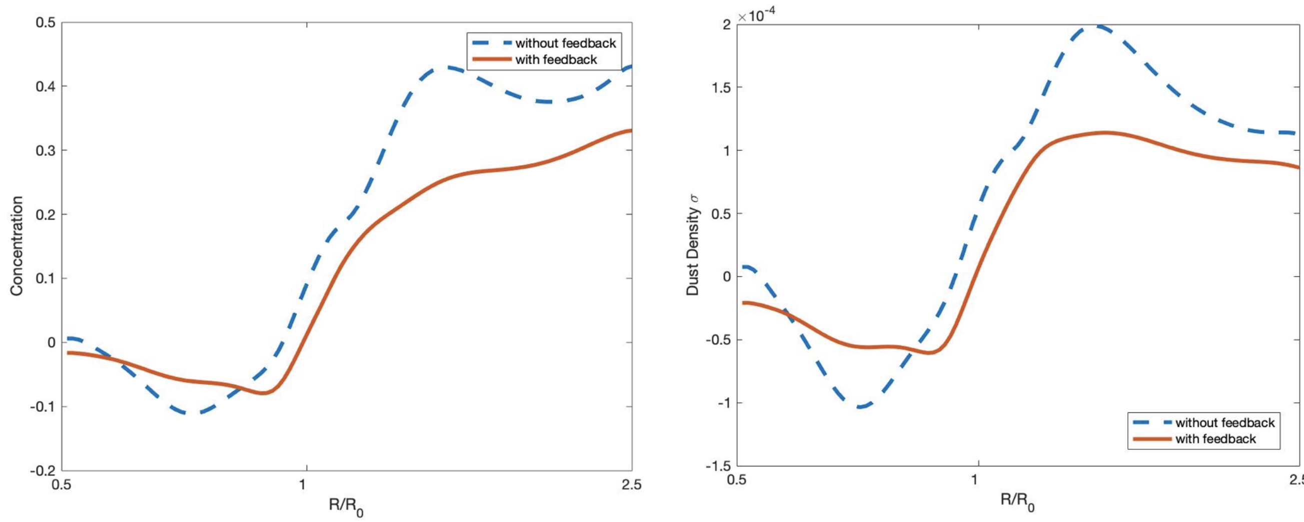

Figure 6: for typical 1cm size objects, comparison of the w/o feedback simulations both after a scale of 10000 time units shows then flattening of dust density in the barrier. Feedback: u_g linearly change through time from $\dfrac{u_g + 2v_g S_t}{1 + S_t^2}$ to zero.

This is also consistent with some previous simulations - in fact, if a planet begins to migrate inwards pushed by the barrier, this flattening effect might soon slow it down so it justifies our emphasis on in situ accretion (Kanagawa 2018, Kanagawa & Kazuhiro 2019).

3 Influence and future prospects

3.1 Opacity and Cooling of Planets

We have deduced in section 1 that the in situ forming of super-Earth may correspond to a halted accretion rate. This may be explained from the opacity parameter.

One of the important factors in determining accretion speed is opacity. Opacity is mainly contributed by dust, grains and heavy metal elements with high average electron number.

Generally, opacity's relation with metallicity is often parameterized as

$$\kappa = \kappa_0 \left(\rho/\rho_0\right)^{\alpha} \left(T/T_0\right)^{\beta} \left(Z/Z_0\right)^{\delta} \tag{17}$$

When the atmosphere is quite "dusty" - assuming the ISM-like grain size distribution of Ferguson et al. (2005), with metals taking the form of dust, metallicity also scales with dust density, so opacity can also be written as:

$$\kappa = 2F_{\kappa}(\rho)\left(\frac{T}{100\mathrm{K}}\right)^{\beta} \mathrm{cm}^2\mathrm{g}^{-1} \tag{18}$$

Where in normal cases $\beta \approx 2$.

And the accretion rate of planet could be given by equaling the accretion luminosity with cooling luminosity(Lee & Chiang. 2015), which means that "to cool is to accrete". As more dust gather around, the energy transfer is hindered and it's more difficult for the heat to get out. With more metallicity taken dust form, the accretion timescale before runaway increases.

Most of previous analysis assumed the F_{κ} to be a constant. So if an in situ accreting core wants to avoid becoming a gas giant due to high opacity, it might have to form somewhere the dust density is originally extraordinarily low. But with grain diffusion in mind, we can look at it from an evolutionist's approach. Maybe even with a very large metallicity and dust density to provide very high opacity, there's this isolation process that we cannot ignore, and the planet core has the ability naturally to transform this dust profile, it might dissipate the high opacity elements and reduce the opacity at this spot to enhance its own radiation and accretion, so it's actually hard for individual super-Earths to avoid growing into gas giants.

But when we are considering a continuous distribution of dust, for example MRN (Mathis et al. 1977), then the original density of the different size particles will yield to some power law of $n(s_d) \propto s_d^{-3.5}$, so the original density $\rho \propto n(s_d){s_d}^3 \propto s_d^{-0.5}$ decreases with particle size and should be multiplied by a factor.

This brings into question the effectiveness of reducing opacity: since the opacity is actually mostly generated by the smaller and denser dust particles, and while pebbles are totally blocked, smaller particles tend to somewhat "slip through" the blockade, we are not totally sure that the reduction of dust density be enough to reduce the runaway timescale. In fact, if the pebbles accumulating inside the pileup collide with each other, disintegrate and generate *additional* small dust particles to slip through, the total opacity might even rise in the most drastic circumstances.

There's another implication on the formation of super-Earths, if we consider the forming of another planet. We can imagine that, as soon as a planet core starts to isolate the pebbles and create this kind of dust profile, the location of becomes a potential breeding ground for super-earths. If another core is formed here, it would have a considerably high opacity naturally, so before it could grow big enough to further push this pile of accumulated grains even further outwards, its accretion would still be hindered by the opacity, and a core would be easier to remain a super-Earth. Vice versa, another planet core formed *closer* to the sun where dust had been evacuated might have very quick accretion rates.

3.2 Planetesimal Formation in Barrier

In fact, inside this dust-accumulating pileup/barrier, it might not only be easier for a core to grow into a super-Earth, but also easier to form the core itself.

Usually, when the dust-to-gas ratio reaches the order of 0.01 it generates streaming instabilities within itself, when it reaches 1 there will be a rapid rise in the growth rate - plausible mechanism

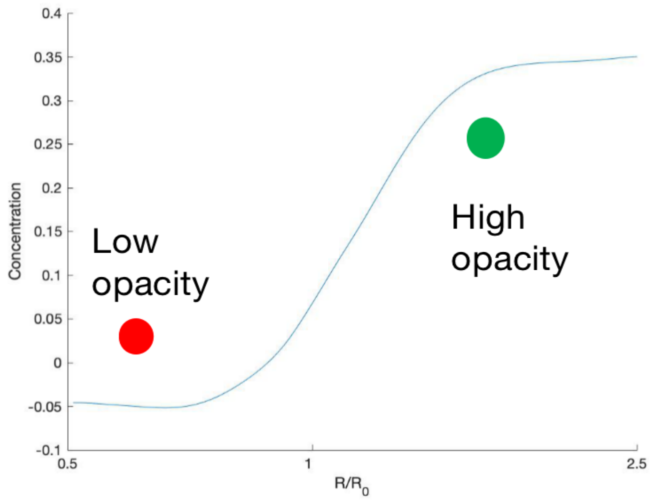

Figure 7: A sketch of our hypotheses: the pebble/dust isolation would create a gap and generate a barrier, the opacity might be reduced inside the gap where the accreting core (red) is located quicken the accretion unless in the condition that the collision of greater dust particles significantly enriches the smaller dust particles. Meanwhile, a new super earth (green) might form inside the accumulating pileup.

for planetesimal formation (Youdin & Goodman 2005). Considering the flattening effects of section 2.4, we can see that if the critical concentration is much greater than 1, then the grains would be able to accumulate enough so that planetesimals could form quickly out of it even if it flattens. If it isn't, then the grains would reach a "ceiling" and flatten out before generating instability. But we can see that for rough calculations, we can bring equation (10) and (17) into (16), and it shows that given enough time C_{crit} should be on the order of unity in almost all the places. Therefore, the instability is very likely to happen.

So there may also be some kind of chain reaction as cores could form out of the pileup one after another and push the pileup further back, leading to the *sequential* formation of a string of super-Earths or close-in super-Earths like the seven earth-sized planets in the TRAPPIST-1 system all located at 0.1AU (Gillon et al. 2016, 2017). The pebble isolation mass (as the criterion for runaway in the pebble accretion paradigm) changes little over such a small distance, so it's very likely that this planets forms out of very similar mechanisms.

3.3 Future Prospects

This paper has laid out the foundation and planned out the detailed routes of our next steps of quantitive analysis: We will use detailed two layer planet accretion models to numerically calculate how a lowering (planet location) or rising (barrier location) dust density induced by pebble accretion might affect the overall opacity and in turn, the runaway timescale; Then, we will use MHD codes to simulate how a dust barrier might lead to planetesimal formation and accretion via streaming instabilities, forming more planet cores and even sequential super-Earths. We would also want to investigate if under certain circumstances the collisions and breaking up of larger particles might actually lead to an increase of opacity at the site of the accreting planet.

Appendix

Jacobi Iteration with MATLAB

We divide our solution C, function of R,t into a matrix, given the initial condition on one side and boundary condition on both sides. We can compare the result of this calculation to the analytical solution for the green function initial condition and fixed boundary condition for both sides to test the validness of this method.

$$R_i = R_{min}(\frac{R_{max}}{R_{min}})^{\frac{i}{100}} := R_{min} r^{\frac{i}{100}} \tag{19}$$

$$\epsilon := r^{\frac{1}{100}} - r^{\frac{-1}{100}} \tag{20}$$

$$\epsilon' := r^{\frac{1}{200}} - r^{\frac{-1}{200}} \tag{21}$$

$$\frac{\partial C}{\partial R}_i = \frac{C_{i+1} - C_{i-1}}{R_{i+1} - R_{i-1}} = \frac{C_{i+1} - C_{i-1}}{R_i \epsilon} \tag{22}$$

While in second order derivatives, we will have to use

$$\frac{\partial C}{\partial R}_i = \frac{C_{i+\frac{1}{2}} - C_{i-\frac{1}{2}}}{R_i \epsilon'} \tag{23}$$

$$\frac{\partial^2 C}{\partial R^2} = \frac{\partial_R C_{i+\frac{1}{2}} - \partial_R C_{i-\frac{1}{2}}}{R_{i+\frac{1}{2}} - R_{i-\frac{1}{2}}} = \frac{\frac{C_{i+1} - C_i}{R_{i+\frac{1}{2}}\epsilon'} - \frac{C_i - C_{i-1}}{R_{i-\frac{1}{2}}\epsilon'}}{R_i \epsilon'} \tag{24}$$

$$\frac{\partial^2 C}{\partial R^2} = \frac{[C_{i+1} + r^{1/100} C_{i-1} - (1 + r^{1/100}) C_i] r^{\frac{-1-4i}{200}}}{R_{min}^2 \epsilon'^2} \tag{25}$$

While linear spacing would be like:

$$\frac{\partial C}{\partial R}_i = \frac{C_{i+1} - C_{i-1}}{2d}, \frac{\partial^2 C}{\partial R^2}_i = \frac{C_{i+1} - 2C_i + C_{i-1}}{d}, d := \frac{R_{max}}{R_{min}}/100 \tag{26}$$

Which gives out hydrodynamic instabilities.

The spacing of t could still be linear taking use of derivative in only one direction:

$$\frac{\partial C}{\partial t}_{i,j} = \frac{C\left(R_i, t_j\right) - C\left(R_i, t_{j-1}\right)}{d} \tag{27}$$

With d as some fixed time step, in this case d=100 time units in FARGO.

So a linear differential equation

$$\frac{\partial C}{\partial t} + A_1(R)\frac{\partial C}{\partial R} + RA_2(R)\frac{\partial^2 C}{\partial R^2} = 0 \tag{28}$$

Could be written in iteration form as:

$$\frac{A_1 d e^{0.0321888i}(0.00402338 C(i+1,j) - 0.00402338\text{C(i-1, j)}) + A_2 d e^{0.0321888i}(0.504024\text{C(i-1, j)} + 0.495976\text{C(i+1, j)}) - 0.0000647566\text{C(i, j-1)} e^{0.0482831i}}{A_2 d e^{0.0321888i} - 0.0000647566 e^{0.0482831i}} \tag{29}$$

In this case equation (10):

$$A_2 = -\kappa_0, A_1 = (u - \kappa_0(2 + \frac{\partial \ln \Sigma}{\partial \ln R}(R))) \tag{30}$$

And modified equations considering gas drag and feedback would be transformed similarly.

Verification with Analytical Answers

We can verify this method with analytical solutions. Clarke & Pringle (1988) gave analytical solution of equation 5 for fixed boundary condition $C|_{bound} = 0$ and Delta function initial condition $C|_{t=0} = C_0\delta(R - R_0)$, in a steady accretion disk of sigma slope $\alpha = 2$, as:

$$C(R,t) = \frac{C_0}{R_0}\left(4\pi\zeta\nu_0 t\right)^{-1/2} \exp\left\{-\frac{\left[\ln\left(R/R_0\right) + 3\nu_0 t\right]^2}{4\zeta\nu_0 t}\right\} \tag{31}$$

By plotting out both the analytical solution and the numerical solution given by our method (FIGURE 8), we can see that the results are reliable.

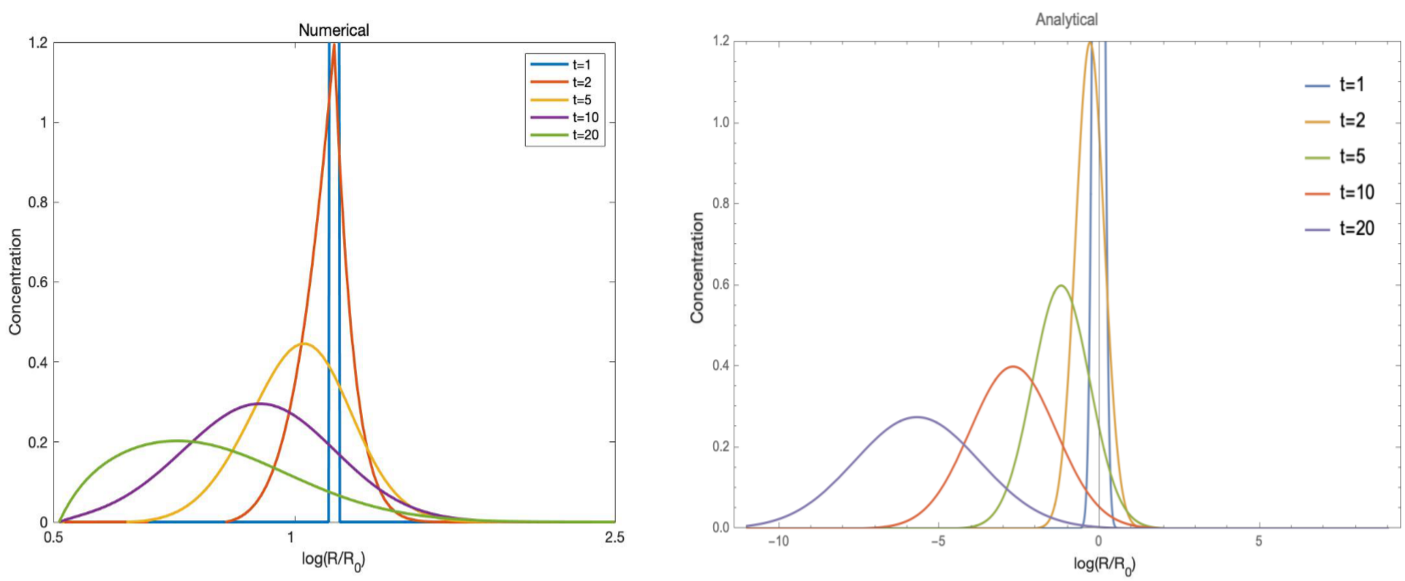

Figure 8: The evolution of concentration over time rewritten with time unit d=1: comparison of analytical results with numerical ones, taking logarithmic spacing on x direction and linear spacing in concentration profile. For a Green function initial, the dust quickly diffuses with its bulk being accreted inwards.

Acknowledgements